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이학박사 학위논문

A fast-slow dynamical systems theory for flocking and synchronization

(플로킹 및 동기화 모델들의 패스트-슬로우 역학계 이론)

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정성은

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A fast-slow dynamical systems theory for flocking and synchronization

A dissertation
submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
to the faculty of the Graduate School of
Seoul National University

by

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Abstract

A fast-slow dynamical systems theory for flocking and synchronization

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In this thesis, we study about the flocking and synchronization models. In specially, we deal with Kuramoto model, Kuramoto model with inertia, Cucker-Smale type model and Newtonian type model with Rayleigh friction. We employ Artstein-Kevrekidis-Slemrod-Titi's unified approach for the singular perturbation to derive a qualitative description of the dynamics in the singular limit. In chapter 2, we review AKST's singular perturbation theory and planar Poincare-Bendixson theory. In chapter 3, we discuss the asymptotic formation of phase-locked states arising from the ensemble of non-identical Kuramoto oscillators. In the formation process of phase-locked states, we estimate the number of collisions between oscillators, and lower-upper bounds of the transversal phase differences. In chapter 4, we present a fast-slow dynamical systems theory for a Kuramoto type model with inertia. In our new formation, order parameters serve as orthogonal observables in the framework of AKST's theory of singular perturbation. In chapter 5, we discuss fast-slow dynamic of planar particle models (Cucker-Smale type model and Newtonian type model with Rayleigh friction).Our analysis employs minimal assumptions on the communication weight. In chapter 6, we briefly present a mathematical problems related by the flocking models and summary this thesis.

Key words: Fast-Slow dynamical system, Kuramoto oscillators, The Cucker-Smale system, focking, synchronization.
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Contents

Chapter 1

Introduction

Emergent flocking phenomena and synchronization appearing in many biological systems are simple collective modes of complex systems, and have been an active research area in applied mathematics, biology and physics, e.g. [4, 24, 25, 26, 60, 71]. The research on the flocking and synchronization using a simple dynamical system was pioneered by Viscek and his collaborators in [68]. Recently several particle models [11, 12, 19, 20, 27, 28, 49, 50, 51, 64, 68] based on continuous-time dynamical systems were employed in the modeling of collective phenomena of biological complex systems. Among them, our interest lies in four particle models. That is, the Kuramoto model, the Kuramoto model with oscillators, the Cucker-Smale type model and the Newtonian type model with Rayleigh friction. The purpose of this paper is to extend the previous results about the above models in some sense. In this process, We employ Artstein-Kevrekidis-Slemrod-Titi's unified approach for the singular perturbation to derive a qualitative description of the dynamics in the singular limit.

The first model we will consider is Kuramoto model which is a prototype for synchronization. Kuramoto oscillators can be visualized as point particles rotating on the unit circle. we establish the existence of phase-locked states arising from the ensemble of non-identical Kuramoto oscillators. The literature most closely related to our problem are that of Chopra-Spong [18], and Ha-Ha-Kim [31]. Both papers basically deal with the complete-frequency synchronization estimates under a priori assumptions that, in the finite time, phase configurations are already in the stability regime which is a subset of the set whose diameter(maximum distance between the particles) is less than $\frac{\pi}{2}$. Hence for initial phase configurations whose diameters are larger than $\frac{\pi}{2}$ the aforementioned results are not directly applicable. However numerical simulation(e.g. Figure 1.(a)) shows that even if the diameter of initial phase configurations is strictly larger than $\frac{\pi}{2}$, the size of phase-diameter first shrinks below a certain value in $\left(0, \frac{\pi}{2}\right)$ determined by initial phase diameter in a finite-time (transition stage), and then tends to the phase-locked state (relaxation stage). So far, the aforementioned literature of Chopra-Spong [18] and Ha-Ha-Kim [31] only deals with the second relaxation stage. In this paper, we explicitly describe the dynamics of the Kuramoto model in the transition stage. More precisely, when initial (phase)-configurations are distributed over an arc whose length is strictly larger than $\frac{\pi}{2}$ and the coupling strength is larger than a certain value, we show that the phase configurations are attracted into a stable regime in a finite-time. Moreover, we present an explicit number of collisions and lower-upper bounds of the transversal phase differences during the relaxation process. During the relaxation process, identical oscillators will be merged together to build non-trivial clusters and non-identical oscillators will be ordered by their natural frequencies.

The second model is Kuramoto model with inertia. This model was first introduced by Ermentrout [29] for modeling the slow relaxation to the phase-locked states among fireflies, Pteroptyx malaccae (see [14] for rigorous justification). However this model also appears in some mechanical models that describe an array of superconducting Josephson junctions [6, 23, 56, 69, 70, 73, 74]. This model exhibits richer phenomena such as a discontinuous first-order phase transition and hysteresis etc. [1, 2, 6, 22, 40, 42, 65, 66] compared with the Kuramoto model without inertia. The purpose of this part is to extend the study of fast-slow dynamical systems theory for synchronization models, which started with [43, 44]. The self-consistent mean-field approach [45, 46, 47] initiated by Kuramoto uses the real order parameters r and ϕ to measure the degree of synchronization:

$$re^{i\phi} := \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}, \quad t \geq 0. \tag{1.0.1}$$

Note that the real order parameters r and ϕ are functions of all phases $\theta_i,\ i = 1, \cdots, N$, which are solutions of the Kuramoto model with inertia. Thus, they depend on K, N, and t implicitly, i.e., $r = r(K, t, N),\ \phi = \phi(K, t, N)$. In Kuramoto's derivation of the critical coupling strength K_c, he assumed that during the double limiting process:

$$t \to \infty \quad \text{and} \quad N \to \infty,$$

the order parameters r and ϕ approached to constant states, which is not obvious a priori. In this paper, we discuss a rigorous mathematical underpinning of Kuramoto's guess of the constant order parameters r and ϕ in an asymptotic limit via some Kuramoto type model with inertia using the framework of Artstein-Kevrekidis-Slemrod-Titi's unified theory (AKST's theory) for a singular perturbation. We prove that the slow motion is just that r and ϕ are constants, whereas the fast motion is an uncoupled pendulum motion given constant torque and unit damping.

The third model is a Cucker-Smale (in short C-S) type model which is a simple dynamical system adopting the philosophy of Newton with a linear velocity coupling as a forcing mechanism. In this paper, we revisit the C-S type flocking model in [41]. In [41], the authors use an assumption that the communication weight has a positive constant lower bound. On the other hand, under the minimal assumption of positive communication weights, we can still establish the translation model observed in flocking of biological system.

The last model is Newtonian type model with the Rayleigh friction as given by Chuang et al [11], D'Orsogna et al [27], Dunkel et al [28] and Schweitzer et al [58]. Unlike the C-S model, this model can describe the milling phenomena [12, 51, 58] often found in the swarming group of fish, insects, etc. The purpose of this part is to rigorous approach the flocking and swarming estimates for this model as a singular perturbation limit.

This thesis has six sections after the introduction. In Section2, we introduce some definition about the synchronization and briefly review fast-slow systems theory and AKST's theory of singular perturbation. In Section3, we provide an improved relaxation process toward the phase-locked states for the non-identical Kuramoto oscillators and quantitative estimates on the exact collision numbers in the relaxation process and lower-upper bounds of the transversal phase differences. In Section 4, we derive a fast-slow dynamical systems formulation of a Kuramoto type model with inertia and a small parameter $\varepsilon > 0$ and then we study the slow motion of the order parameters in the limit $\varepsilon \to 0$ using AKST's theory. In Section5, we consider fast-slow dynamics of planar models, a Cucker-Smale type flocking model and a Newtonian type particle model in the singular perturbation limit. Finally, Section 6 provide the summary of this thesis.

Chapter 2

Preliminaries

In this preliminary chapter, let me introduce exact form of models mentioned in the introduction. Also we review invariant measures, Young measures and Artstein-Kevrekedis-Slemrod-Titi's unified approach to singular perturbations [55, 57, 63]. A similar presentation has been given in [36] but we include it here to keep this article self-contained. The AKST theory provides a generalization of classical averaging [57] and Tikhonov [63] theory and the basic ideas were first presented by Artstein and Vigodner [7] with a subsequent survey by Artstein in [5].

2.1 Flocking and synchronization models

First of all, we introduce exact form of flocking and synchronization models.

- **Kuramoto oscillators**

Let x_i be the position of the i-th rotor and we rewrite $x_i = e^{\sqrt{-1}\theta_i}$ via polar coordinates. The dynamics of Kuramoto oscillators can be effectively governed by the following phase model:

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i), \quad t > 0, \quad i = 1, \dots, N,$$

subject to initial data:

$$\theta_i(0) = \theta_i^0,$$

where K is the uniform positive coupling strength, and Ω_i represents the intrinsic natural frequency of the i-th oscillator drawn from some distribution function $g = g(\Omega)$. The explicit form of g is irrelevant in the following analysis.

- **Kuramoto oscillators with inertia**

The Kuramoto model with inertia reads as follows:

$$m\ddot{\theta}_i + \dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i), \quad t > 0, \quad i = 1, \dots, N,$$

subject to the initial data:

$$(\theta_i, \dot{\theta}_i)\Big|_{t=0} = (\theta_{i0}, \omega_{i0}),$$

where m is constant inertia.

- **Cucker-Smale type model**

Let $(x_i(t), v_i(t)) \in \mathbb{R}^{2d}$ be the phase-space coordinate of the i-th particle at time t, and its dynamics is governed by the following singularly perturbed ODE system: For $i = 1, \cdots, N$,

$$\begin{aligned}
\frac{dx_i}{dt} &= v_i, \quad t > 0, \\
\frac{dv_i}{dt} &= \frac{K}{\varepsilon N}\sum_{j=1}^{N} \psi(||x_j - x_i||)(v_j - v_i) + g_i(v),
\end{aligned}$$

subject to initial data:

$$(x_i, v_i)(0) = (x_{i0}, v_{i0}).$$

Here K is a positive constant representing the strength of coupling and g_i may incorporate friction forces, and ψ is a communication weight between particles. In this and next models, the small parameter ε is the mass of the particle. In very simplistic terms, we may say that we are studying the flocking and swarming of bacteria, birds and bees and not whales or elephants.

- **Newtonian type model with the Rayleigh friction**

This model is the following particle model on $\mathbb{R}^2$: For $i = 1, \cdots, N$,

$$\begin{aligned} \frac{dx_i}{dt} &= v_i \\ \varepsilon \frac{dv_i}{dt} &= -\sum_{j \neq i} \nabla_x \varphi(x_j - x_i) + \delta(1 - ||v_i||^2) v_i \end{aligned}$$

subject to initial data:

$$(x_i, v_i)(0) = (x_{i0}, v_{i0}),$$

where φ is a pairwise interaction potential and δ is a positive constant relating the strength of self-accelerating force and self-decelerating force, respectively.

2.2 A fast-slow dynamical system

2.2.1 Invariant measures and Young measures

In this part, we collect some basic notions from [3] on the invariant measures and Young measures to be used in later sections. For detailed discussions, we refer to [8, 54, 67].

Recall that a probability measure μ on $\mathbb{R}^N$ is a σ-additive set function defined on the Borel subsets of $\mathbb{R}^N$ with values in $[0,1]$ and $\mu(\mathbb{R}^N) = 1$. We set $\mathcal{P}(\mathbb{R}^N)$ to be the family of all probability measures on $\mathbb{R}^N$ endowed with weak convergence of measures,see Billingsley [9].

Definition 2.2.1. *Let μ_n be the sequence of probability measures in $\mathbb{R}^N$. Then μ_n weak convergent to some probability measure μ is defined by following:*

$$\int_{\mathbb{R}^N} f d\mu_n \to \int_{\mathbb{R}^N} f d\mu, \quad f : \textit{continuous and bounded on } \mathbb{R}^N$$

Definition 2.2.2. *Let μ be a probability measures defined on $\mathbb{R}^N$.*

1. *The support of μ (often denoted by spt(μ)) is the smallest closed set C such that $\mu(C) = 1$.*

2. *μ is an invariant measure of the system*

$$\frac{dx}{dt} = f(x), \quad f \text{ : Lipshitz continuous,} \tag{2.2.1}$$

if the solutions $X(t, x_0)$ to (2.2.1) for x_0 in a neighborhood of spt(μ) are defined on a common interval I around $t = 0$ and if $\mu(B) = \mu(X(t, B))$ for each $t \in I$ and every Borel set B.

3. *μ is a Young measure if $\mu(\cdot)\ : [a, b] \to \mathcal{P}(\mathbb{R}^N)$ is a measurable map.*

The Young measure $\mu(\cdot)$ can be identified with the measure on $[a, b] \times \mathbb{R}^N$ obtained via integration,

$$\bar{\mu}(C \times B) = \int_C \mu(t)(B)dt.$$

So it is make sense to equate the convergence between the young measures μ_n with the convergence of $\bar{\mu}_n$. Because of the following relation, we can obtain the definition of convergence of young measures.

$$\begin{aligned}
\mu_j \to \mu_0 \quad &\Leftrightarrow \quad \bar{\mu}_j \to \bar{\mu}_0 \\
&\Leftrightarrow \quad \int_a^b \int_{\mathbb{R}^N} h(t, x) d\bar{\mu}_j \to \int_a^b \int_{\mathbb{R}^N} h(t, x) d\bar{\mu}_0 \\
&\Leftrightarrow \quad \int_a^b \int_{\mathbb{R}^N} h(t, x)\mu_j(t)(dx)(dt) \to \int_a^b \int_{\mathbb{R}^N} h(t, x)\mu_0(t)(dx)(dt)
\end{aligned}$$

Definition 2.2.3. [9] *Let (μ_j) be a sequence of Young measures defined on the same interval $[a, b]$. The sequence μ_j converges to the Young measure μ_0 if and only if*

$$\int_a^b \int_{\mathbb{R}^N} h(x, t)\mu_j(t)(dx)dt \to \int_a^b \int_{\mathbb{R}^N} h(x, t)\mu_0(t)(dx)dt,$$

for every continuous and bounded real function $h = h(x, t)$.

Remark 2.2.1. *1. The continuity of a test function h in time-variable can be replaced by measurability.*

2. A usual point-valued function $x = x(\cdot)$ can be viewed as a Young measure, when the point $x(t)$ is identified with the Dirac measure supported on the singleton $\{x(t)\}$. Hence when we refer to the convergence of a sequence of functions in the sense of Young measures, we mean the convergence in the sense of Definition 2.2 for the corresponding Dirac measure-valued maps. Thus when we have a sequence of continuous functions uniformly bounded in j, $\{x_j(t)\}$, $a \leq t \leq b$, its associated Young measures are $\mu_j(t) = \delta(x - x_j(t))$. If we choose $h(x,t) = a(x)b(t)$, then convergence of $x_j(\cdot)$ to the Young measure μ_0 in Young measures implies the statement

$$\int_a^b b(t)a(x_j(t))dt \to \int_a^b b(t)\Big(\int_{\mathbb{R}^N} a(x)\mu_0(t)(dx)\Big)dt.$$

Hence the weak-$*$ $L_\infty([a,b])$ limit of the functions $a(x_j(\cdot))$ is represented by the value

$$\int_{\mathbb{R}^N} a(x)\mu_0(t)(dx).$$

This representation has proved extremely valuable in applications [62].

2.2.2 Review on AKST's unified approach

Consider a fast-slow system:

$$\frac{dU}{dt} = \frac{F(U)}{\varepsilon} + G(U), \quad U \in \mathbb{R}^N, \quad t > 0, \tag{2.2.2}$$

subject to initial data:

$$U(0) = U_0, \quad t = 0, \tag{2.2.3}$$

where $F(U), G(U) : \mathbb{R}^N \to \mathbb{R}^N$ are continuous functions and denote fast and slow parts of the system (2.2.2) respectively.

We first present a framework of AKST's theory:

- ($\mathcal{F}1$): The solutions of (2.2.2) lie in a compact set $H \subset \mathbb{R}^N$ on some interval, say $0 \leq t \leq 1$ for any $0 < \varepsilon \ll 1$ and in addition, there is a compact set $K \subset H$ which is positively invariant with respect to the fast part of (2.2.2):

$$\frac{dU}{ds} = F(U). \tag{2.2.4}$$

- ($\mathcal{F}2$): For initial data $U_0 \in \mathrm{K}$ solutions of the full system (2.2.2) and fast system (2.2.4) are unique.

Theorem 2.2.1. *Let U^ε be solutions of (2.2.2) satisfying $U_0^\varepsilon = U_0 \in K$, and defined on a common interval, say $[0,1]$. Then for every sequence $\varepsilon_j \to 0$, there exists a subsequence $U^{\varepsilon_j}(\cdot)$ which converges in the sense of Young measures to a Young measure, say $\mu_0(\cdot)$ defined on $[0,1]$. The value of the limit Young measure is an invariant measure for the fast equation (2.2.4).*

Definition 2.2.4. *Let $V : \mathbb{R}^N \to \mathbb{R}$ be a continuous function called a measurement.*

(i) For a given probability measure μ, we call the action of V on a measure μ:

$$\hat{V}(\mu) := \int_{\mathbb{R}^N} V(\lambda)\mu(d\lambda),$$

an observable.

(ii) The observable $\hat{V}(\mu)$ is an "orthogonal observable" of the fast-part (2.2.4), if the measurement $V(\cdot)$ is a first integral of the fast system (2.2.4), i.e., $V(U(s))$ is constant along any solution of (2.2.4) and hence is equivalent to the relation $\nabla_U V \cdot F \equiv 0$ if V is differentiable. Here $a \cdot b$ is the standard Euclidean inner product of two vectors $a, b \in \mathbb{R}^N$.

Theorem 2.2.2. *Suppose the assumptions $(\mathcal{F}1) - (\mathcal{F}2)$ hold, and let $U^{\varepsilon_j}(\cdot)$ be the solutions of (2.2.2) satisfying $U^{\varepsilon_j}(0) = U_0$ and defined on, say $[0,1]$, and which converge to $\mu_0(\cdot)$ in the sense of Young measures via Theorem 2.1. Then for any orthogonal observable $\hat{V}(\cdot)$ of the system (2.2.2), the measurements $V(U^{\varepsilon_j}(t))$ converge in weak-* $L_\infty([0,1])$ to $\hat{V}(\mu_0(t))$:*

$$\hat{V}(\mu_0(t)) = \int_{\mathbb{R}^N} V(\lambda)\mu_0(t)(d\lambda).$$

Moreover, $\hat{V}(\mu_0(t))$ satisfies the relation:

$$\hat{V}(\mu_0(t)) = V(U_0) + \int_0^t \int_{\mathbb{R}^N} \nabla V(\lambda) \cdot G(\lambda)\mu_0(s)(d\lambda)ds. \tag{2.2.5}$$

Remark 2.2.2. *One may ask why we do not differentiate the above integral relation to obtain an ordinary differential equation for $\hat{V}(\mu_0(t))$. First we note even if we could differentiate the integral relation, it would not yield an ordinary differential equation in the classical sense, i.e., since the Young measure $\mu_0(t)$ is determined via the initial data U_0, the right hand side depends on the initial data. Secondly the issue of differentiability has been covered in Theorem 6.5 of [3]. The sufficient conditions given there are that the Young measure μ_0 is uniquely determined by the initial data U_0 and furthermore, that it is Lipschitz continuous as a function of the data U_0.*

2.2.3 Limit dynamics of a planar dynamical system

In this part, we briefly discuss the Poincare-Bendixson theorem for a planar system. We follow the notation and presentation given in Coddington and Levinson [15].

Consider a two-dimensional dynamical system:

$$\begin{aligned} \frac{dx_1}{dt} &= f_1(x_1, x_2), \quad t > t_0, \\ \frac{dx_2}{dt} &= f_2(x_1, x_2), \end{aligned} \tag{2.2.6}$$

subject to initial data

$$(x_1, x_2)(t_0) = (x_{10}, x_{20}). \tag{2.2.7}$$

For simplicity, we set

$$x_0 := (x_{10}, x_{20}) \quad \text{and} \quad x(t) := (x_1(t), x_2(t)).$$

Definition 2.2.5. *Let (x_1, x_2) be the solution to the system*

1. *The time-forward orbit $\mathcal{C}^+$ and its corresponding ω-limit $\omega(x_0)$ are defined as follows.*

$$\begin{aligned} \mathcal{C}^+(x_0) &:= \{x(t) \in \mathbb{R}^2 : \ t \geq t_0\}, \\ \omega(x_0) &:= \textit{set of all limit points of } \mathcal{C}^+. \end{aligned}$$

2. *Let P be an isolated equilibrium point for the system. Then the index of P is defined as the index of any Jordan curve containing only P among all equilibria in its interior.*

Remark 2.2.3. *1.The index of a saddle point is -1 and the indices of other types of equilibria are 1.*
2. A bounded ω-limit set is connected.

The following theorem classifies all possible structures of ω-limit sets.

Theorem 2.2.3. *Let $\mathcal{D}$ be a bounded open subset of $\mathbb{R}^2$, and assume that the system (2.2.6) has a finite number of equilibrium points. If $\mathcal{C}^+$ is contained in a closed subset K of D, then the following trichotomy holds.*

1. *$\omega(x_0)$ consists of single equilibrium point of (2.2.6).*
2. *$\omega(x_0)$ is a periodic orbit.*
3. *$\omega(x_0)$ consists of a finite number of equilibrium points of (2.2.4), and a set of orbits connecting equilibrium points. In particular, if the orbit is periodic, the sum of all indices of equilibria contained in the periodic orbit is exactly one.*

Chapter 3

Asymptotic formation of phase-locked states for the Kuramoto oscillators

3.1 Definitions and Motivating problems

In this section, we briefly review relevant results on the formation of phase-locked states for the Kuramoto oscillators in [18, 31] and provide two motivating questions to be covered in this chapter. Recall that the Kuramoto oscillator is following phase model:

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \quad t > 0, \quad i = 1, \dots, N, \tag{3.1.1}$$

subject to initial data:

$$\theta_i(0) = \theta_i^0, \tag{3.1.2}$$

We introduce averaged variables as follows.

$$\theta_c := \frac{1}{N}\sum_{i=1}^{N}\theta_i, \quad \omega_c := \frac{1}{N}\sum_{i=1}^{N}\omega_i, \quad \Omega_c := \frac{1}{N}\sum_{i=1}^{N}\Omega_i,$$

where $\omega_i := \dot{\theta}_i$ is the instantaneous frequency of the i-th oscillator.

It is easy to see from (3.1.1) that the averaged phase rotates the unit circle with a constant averaged natural frequency Ω_c:

$$\frac{d\theta_c}{dt} = \Omega_c, \quad \text{i.e.,} \quad \theta_c(t) = \theta_c(0) + t\Omega_c, \quad t \geq 0.$$

On the other hand, the fluctuations $(\hat{\theta}_i, \hat{\Omega}_i) := (\theta_i - \theta_c, \Omega_i - \Omega_c)$ satisfy the same form of equation:

$$\dot{\hat{\theta}}_i = \hat{\omega}_i = \hat{\Omega}_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\hat{\theta}_j - \hat{\theta}_i), \tag{3.1.3}$$

with the additional algebraic constraints:

$$\sum_{i=1}^{N} \hat{\theta}_i = 0, \quad \sum_{i=1}^{N} \hat{\Omega}_i = 0.$$

We next recall the definitions of several terminologies to used throughout the paper.

Definition 3.1.1. *Let $\theta = \theta(t)$ be any global smooth solution to the Kuramoto model* (3.1.1).

1. *The Kuramoto model exhibits asymptotic complete-phase synchronization if and only if the relative phase differences go to zero asymptotically:*
$$\lim_{t\to\infty} |\theta_i(t) - \theta_j(t)| = 0, \qquad \forall\ i \neq j.$$
2. *The Kuramoto model exhibits asymptotic complete-frequency synchronization if and only if the relative frequency differences go to zero asymptotically:*
$$\lim_{t\to\infty} |\omega_i(t) - \omega_j(t)| = 0, \qquad \forall\ i \neq j.$$
3. *The dynamical state $\theta(t) = (\theta_1(t), \cdots, \theta_N(t))$ is called an asymptotically phase-locked state for the Kuramoto model* (3.1.1) *if and only if each relative phase differences go to the constant as $t \to \infty$, i.e.,*
$$\lim_{t\to\infty} |\theta_i(t) - \theta_j(t)| = \theta_{ij}, \quad \textit{for all} \quad i, j \in \{1, \ldots, N\}. \tag{3.1.4}$$

4. *Two oscillators i and j have a collision at time t if and only if their relative phase difference $\theta_i - \theta_j$ becomes zero at time t:*

$$\theta_i(t) = \theta_j(t).$$

Remark 3.1.1. *1. The condition* (3.1.4) *is equivalent to say*

$$\hat{\omega}_i(t) = \omega_i(t) - \omega_c(t) \to 0, \quad \text{as } t \to \infty.$$

2. In the thermodynamic limit ($N \to \infty$), phase-locked states correspond to the traveling profile moving with the averaged natural frequency Ω_c, i.e., $\varphi(\cdot - \Omega_c t)$, for some $\varphi \in L_1(\mathbb{T})$.

3.1.1 Dynamics of phase diameter in a stable regime

In this part, we briefly discuss main results in [18, 31] on the complete-frequency synchronization. We first introduce several notations to be used throughout the thesis:

$$\mathcal{D}(\theta(t)) := \max_{1\le i,j\le N} |\theta_i(t) - \theta_j(t)|, \ \mathcal{D}(\omega(t)) := \max_{1\le i,j\le N} |\omega_i(t) - \omega_j(t)|,$$
$$\mathcal{D}(\Omega) := \max_{1\le i,j\le N} |\Omega_i - \Omega_j|, \quad K_e := \frac{\mathcal{D}(\Omega)}{\sin \mathcal{D}(\theta^0)},$$

where

$$\omega := (\omega_1, \omega_2, \cdots, \omega_N), \quad \Omega := (\Omega_1, \cdots, \Omega_N).$$

Remark 3.1.2. *Without loss of generality, we may assume that $\mathcal{D}(\theta^0) \neq 0$. If $\mathcal{D}(\theta^0) = 0$, then there exists $t_0 > 0$ such that $\mathcal{D}(\theta(t_0)) > 0$. If not, $\mathcal{D}(\theta(t)) = 0$ for all t. This is impossible because complete-phase synchronization state is not a solution to* (3.1.1) *for non-identical oscillators. Hence we can begin our process from $t = t_0$ again.*

In order to motivate the paper, we quote two recent results on the complete-frequency synchronization in a stable regime [18, 31]. We set $\hat{\mathcal{D}}^\infty$ to be the root of the following trigonometric equation:

$$\sin x = \frac{\mathcal{D}(\Omega)}{K}, \quad x \in \Big(0, \frac{\pi}{2}\Big).$$

Theorem 3.1.1. (Complete synchronization [31]) *The following estimates hold.*
(i) (Identical oscillators): Let $\theta = \theta(t)$ be the global smooth solution to the system (3.1.1)-(3.1.2) *satisfy*

$$\mathcal{D}(\Omega) = 0, \quad K > 0, \quad \mathcal{D}(\theta^0) < \pi.$$

Then for any number η with $0 < \eta \ll 1$, there exists $\tau \gg 1$ such that

$$\mathcal{D}(\theta(t)) \le K\mathcal{D}(\theta(\tau))e^{-K(1-\eta)(t-\tau)}, \quad t \ge \tau.$$

(ii) (Non-identical oscillators): Let $\theta = \theta(t)$ be the global smooth solution to the system (3.1.1)-(3.1.2) *with*

$$\mathcal{D}(\Omega) > 0, \quad K > \mathcal{D}(\Omega), \quad \mathcal{D}(\theta^0) < \hat{\mathcal{D}}^\infty.$$

Then we have

$$\mathcal{D}(\omega(t)) \le \mathcal{D}(\omega(0))e^{-K(\cos\hat{\mathcal{D}}^\infty)t}, \quad t \ge 0.$$

Remark 3.1.3. *For the Kuramoto oscillators with inertia, we refer to [14].*

Theorem 3.1.2. [18] *Let $\theta = \theta(t)$ be the global smooth solution to the system* (3.1.1)-(3.1.2). *If there exists $T > 0$ such that*

$$\mathcal{D}(\theta(t)) < \frac{\pi}{2} - \varepsilon, \quad t \ge T,$$

where $\varepsilon \in (0, \frac{\pi}{2})$. Then the oscillator-frequencies ω_i synchronize exponentially to the mean frequency Ω_c and satisfy

$$\max_{1\le i\le N} |\omega_i - \Omega_c| \le \sigma_T e^{-K(\sin\varepsilon)(t-T)}, \quad \sigma_T > 0, \quad t \ge T.$$

We next discuss some limitations of Theorem 3.1.1 and Theorem 3.1.2. Note that if we introduce a scaled time variable

$$\tau := Kt,$$

then the Kuramoto model (3.1.1) can be rewritten as

$$\frac{d\theta_i}{d\tau} = \frac{\Omega_i}{K} + \frac{1}{N}\sum_{j=1}^N \sin(\theta_j - \theta_i). \tag{3.1.5}$$

Since

$$\lim_{K\to\infty} \frac{\Omega_i}{K} = 0,$$

the system (3.1.5) behaves like the Kuramoto model with $\Omega_i = 0$ for sufficiently large K. Hence for non-identical oscillators, the second estimate in Theorem 3.1.1 should be reduced to the first estimate in Theorem 3.1.1 to be optimal. However, as $K \to \infty$, $\hat{\mathcal{D}}^\infty$ approaches to 0, which shows that the second estimate in Theorem 3.1.1 need to be improved. In fact, we can choose $\hat{\mathcal{D}}^\infty$ in the interval $\left(\frac{\pi}{2}, \pi\right)$, but in this case, since $\cos\hat{\mathcal{D}}^\infty < 0$, the decay estimate in frequency diameter is not guaranteed (see Theorem 3.3 in [31]).

We now return to the discussion of Theorem 3.1.2. For $\varepsilon = 0$, the estimate in Theorem 3.1.2 does not give any decay estimate except uniform boundedness. This is still not optimal as can be seen from Figure 3.1.(a). Based on the above discussion, the natural questions that we can ask are:

- Question A (Formation): For any initial configurations with $\mathcal{D}(\theta^0) < \pi$, for the complete-frequency synchronization, how large K do we need ?
- Question B (Quantitative estimates): In the complete-frequency synchronization, is the number of collisions finite? If so, can we explicitly calculate the number of collisions from the given initial configurations and natural frequencies ?

In the following two sections, we will study the above questions separately.

Before we close this section, we briefly discuss two dynamic stages "transition stage" and "relaxation stage" appearing in the complete frequency synchronization process for the Kuramoto flow (3.1.1). For example, in the complete frequency synchronization process, for an initial phase configuration θ^0 with a phase diameter $D(\theta^0) > \frac{\pi}{2}$, the diameter of frequency configuration begins to shrink with a slower speed until some time $t = t_*$, and then it decays to zero exponentially fast afterwards (see Figure 3.3). Of course, the determination of a sharp transition time t_* between these two stages is not easy to determine, and might be dependent on the initial configuration and coupling strength. Hence we will provide a larger time interval which may include this transition time. For this, we set the dual angle $\mathcal{D}^\infty$ of initial phase diameter $\mathcal{D}(\theta^0)$ by

$$\mathcal{D}^\infty \in \Big(0, \frac{\pi}{2}\Big), \quad \sin \mathcal{D}^\infty = \sin \mathcal{D}(\theta^0).$$

For an initial diameter $\mathcal{D}(\theta^0) \in [0, \frac{\pi}{2}]$, the dual angle $\mathcal{D}^\infty$ coincides with $\mathcal{D}(\theta^0)$. On the other hand, for a given initial phase configuration θ^0 with a diameter $\mathcal{D}(\theta^0) \in \Big(\frac{\pi}{2}, \pi\Big)$, we define $t_*(\theta^0)$ to be the first time satisfying the relation

$$\mathcal{D}(\theta(t_*)) = \mathcal{D}^\infty.$$

In contrast, for the initial phase configuration with $\mathcal{D}(\theta^0) \in [0, \frac{\pi}{2})$, we simply set $t_*(\theta^0) = 0$. It seems that the slower decay occurs in the transition stage $[0, t_0(\theta^0))$ in numerical simulations (see Section 3.4.4), although we cannot prove it rigorously.

3.2 Formation of phase-locked states

In this section, we present basic a priori estimates and a transition estimate from the transition stage to the relaxation stage.

3.2.1 Overview of our strategy

In the literature [18, 31], initial phase-configurations lying in the stable regime were employed in the relaxation process to phase-locked states. Hence to be consistent with numerical simulation(e.g. Figure 3.1.(a)), we need to establish a transition estimate describing the entrance of phase configurations located outside a stable(relaxation) regime to a stable regime in a finite-time. Note that the coupling strength K_e:

$$K_e = \frac{\mathcal{D}(\Omega)}{\sin \mathcal{D}(\theta^0)},$$

is the sufficient strength to push an initial phase configuration toward a stable regime (see Proposition 3.2.1). Hence before phase configurations enter a stable regime, the uniform coupling strength is large enough to push phase configurations until they enter a stable regime and stay there afterward. This is due to the following inequality:

$$\mathcal{D}(\theta(t)) \in [\mathcal{D}^\infty, \mathcal{D}(\theta^0)] \qquad \Longrightarrow \qquad \frac{\mathcal{D}(\Omega)}{\sin \mathcal{D}(\theta(t))} \le \frac{\mathcal{D}(\Omega)}{\sin \mathcal{D}(\theta^0)} := K_e,$$

where $\mathcal{D}^\infty$ is the dual angle of $\mathcal{D}(\theta^0)$ satisfying

$$\mathcal{D}^\infty \in \Big(0, \frac{\pi}{2}\Big), \qquad \sin \mathcal{D}^\infty = \sin \mathcal{D}(\theta^0).$$

In the sequel, we define the stable zone $\mathcal{S}(\theta^0)$ of initial configuration θ^0 as follows

$$\mathcal{S}(\theta^0) := \{\theta \in \mathbb{T}^N \;:\; \mathcal{D}(\theta) \le \mathcal{D}^\infty\}.$$

We next describe the several steps for the complete-frequency synchronization of initial phase-configurations. We basically use so called a bootstrapping argument as heavily used in [14, 31]. For a given initial phase configuration θ^0 satisfying

$$\mathcal{D}(\theta^0) \in \Big(\frac{\pi}{2}, \pi\Big),$$

we will show the formation of phase-locked states in the following steps:

- Step A (Rough estimate): Under the large coupling strength $K > K_e$, we show that the initial phase configuration is non-expansive in the sense that
$$\sup_{t \ge 0} \mathcal{D}(\theta(t)) \le \mathcal{D}(\theta^0).$$

- Step B (Refined estimate): Based on the rough estimate in Step A, we show that the initial configuration enters into its stable zone $\mathcal{S}(\theta^0)$ in a finite-time, and then stays inside the stable zone $\mathcal{S}(\theta^0)$ along the forward Kuramoto flow (3.1.1).

- Step C (Relaxation estimate): We finally apply the complete-frequency synchronization results [18, 31] to the configuration lying inside the stable zone to obtain the asymptotic complete-frequency synchronization.

We now list main result on the formation of phase-locked states in the following theorem.

Theorem 3.2.1. (Formation of phase-locked states) *Let $\theta = \theta(t)$ be the global smooth the solution to the system* (3.1.1)-(3.1.2) *satisfying*

$$0 < \mathcal{D}(\theta^0) < \pi, \quad \mathcal{D}(\Omega) > 0, \quad K > K_e. \tag{3.2.1}$$

Then there exists $t_0 > 0$ such that

$$\mathcal{D}(\omega(t_0))e^{-K(t-t_0)} \leq \mathcal{D}(\omega(t)) \leq \mathcal{D}(\omega(t_0))e^{-K(\cos\mathcal{D}^\infty)(t-t_0)}, \quad t \geq t_0.$$

Remark 3.2.1. *1. For the case of identical oscillators with $\mathcal{D}(\Omega) = 0$, the coupling strength K_e becomes 0, so as long as $K > 0$, Lemma 3.2.1 holds. This is exactly the case of Lemma 3.1 in [31].*

2. The assertion of Lemma 3.2.1 implies that phase configurations lying on the half circle and not in phase-synchronized state become the complete-frequency synchronized states asymptotically, although the result in Lemma 3.2.1 does not tell us the exact asymptotic phase-diameter. Later, in Proposition 3.2.1, we will see that $\mathcal{D}(\theta(t))$ is decreasing to an asymptotic value below $\mathcal{D}^\infty$. We also know that the result of Lemma 3.2.1 does not mean that $\mathcal{D}(\theta(t))$ is a decreasing function for time t. It only shows that the boundedness of $\mathcal{D}(\theta(t))$ by $\mathcal{D}(\theta^0)$.

3. Recently, Dorfler and Bullo revisited the sufficient and necessary conditions for the exponential synchronization to the Kurmoto model [22] using a qualitative analysis.

4. The result of Lemma 3.2.1 improves the main result of [18]. Below, we briefly discuss our result with that of [18]. The invariant set in Lemma 3.2.1 is larger than that of [18]. In fact, the trapping region given in [18] is $\{\theta = (\theta_1, \cdots, \theta_N) \in \mathbb{T}^N : |\theta_i - \theta_j| \leq \frac{\pi}{2} - 2\varepsilon\}$ for $\varepsilon > 0$.

5. Note that our interest in this paper is to find sufficient conditions to guarantee the formation of stable equilibria "phase-locked states" using the elementary analytic methods. However it will be very interesting to consider the case where our sufficient condition (3.2.1) *is violated. In this case, the partial phase-locked states [21] and complicated chaotic states [53] can also emerge* *from initial configurations. Other geometric dynamical system approaches for the stability of phase-locked states such as synchronization and splay states can be found in previous literature [6, 16, 17, 52, 73, 74].*

3.2.2 Basic a priori estimates

Before we present a complete-frequency synchronization estimate, we first show that there exists a positively invariant set for the system (3.1.1)-(3.1.2). For this, we introduce extremal phases θ_M, θ_m and phase diameter $\mathcal{D}(\theta)$:

$$\theta_M := \max_{1\le i\le N} \theta_i, \quad \theta_m := \min_{1\le i\le N} \theta_i, \quad \mathcal{D}(\theta) := \theta_M - \theta_m.$$

Lemma 3.2.1. (Existence of a positively invariant set) *Let $\theta = \theta(t)$ be the global smooth solution to the system* (3.1.1)-(3.1.2) *satisfying*

$$0 < \mathcal{D}(\theta^0) < \pi, \quad \mathcal{D}(\Omega) > 0, \quad K > K_e.$$

Then the phase-diameter $\mathcal{D}(\theta(t))$ is uniformly bounded by $\mathcal{D}(\theta^0)$:

$$\mathcal{D}(\theta(t)) \le \mathcal{D}(\theta^0), \quad t \ge 0.$$

Proof. Note that $\mathcal{D}(\theta(t))$ is Lipschitz continuous and differentiable except the collision times between the extremal phases and other neighboring phases. We use the assumption $K > K_e = \frac{\mathcal{D}(\Omega)}{\sin \mathcal{D}(\theta^0)}$ to obtain

$$\begin{aligned}
&\frac{d^+\mathcal{D}(\theta(t))}{dt}\Big|_{t=0} \\
&= \Omega_M - \Omega_m + \frac{K}{N}\sum_{j=1}^N \Big[\sin(\theta_j^0 - \theta_M^0) - \sin(\theta_j^0 - \theta_m^0)\Big] \\
&\le \mathcal{D}(\Omega) + \frac{K}{N}\frac{\sin\mathcal{D}(\theta^0)}{\mathcal{D}(\theta^0)}\sum_{j=1}^N \Big[(\theta_j^0 - \theta_M^0) - (\theta_j^0 - \theta_m^0)\Big] \\
&= \mathcal{D}(\Omega) - K\sin\mathcal{D}(\theta^0) \\
&< 0,
\end{aligned} \tag{3.2.2}$$

where we used

$$-\mathcal{D}(\theta^0) \le \theta_j^0 - \theta_M^0 \le 0, \quad \sin x \le \frac{\sin\mathcal{D}(\theta^0)}{\mathcal{D}(\theta^0)} x \quad \text{for } x \in [-\mathcal{D}(\theta^0), 0],$$

$$0 \leq \theta_j^0 - \theta_m^0 \leq \mathcal{D}(\theta^0), \quad \sin x \geq \frac{\sin \mathcal{D}(\theta^0)}{\mathcal{D}(\theta^0)} x \quad \text{for } x \in [0, \mathcal{D}(\theta^0)].$$

Therefore $\mathcal{D}(\theta(t))$ starts to strictly decrease at $t = 0+$. To obtain our main result, we need to show that $\mathcal{D}(\theta(t))$ cannot expand its magnitude beyond $\mathcal{D}(\theta^0)$. Suppose $\mathcal{D}(\theta(t))$ reaches the value $\mathcal{D}(\theta^0)$ at $t = t_0$ and $\mathcal{D}(\theta(t)) < \mathcal{D}(\theta^0)$, $t < t_0$. Then at that time, it is easy to see that

$$\frac{d^- \mathcal{D}(\theta(t))}{dt}\Big|_{t=t_0} \leq \mathcal{D}(\Omega) - K \sin \mathcal{D}(\theta(t_0)) = \mathcal{D}(\Omega) - K \sin \mathcal{D}(\theta^0) < 0. \quad (3.2.3)$$

This is a contradiction since $\mathcal{D}(\theta(t))$ has to increase so as to have a value $\mathcal{D}(\theta^0)$ at $t = t_0$. This yields the desired result. □

As a direct corollary of Lemma 3.2.1, we can obtain a nontrivial lower bound of Kuramoto's real order parameter $r = r_N(K, t)$: For a phase configuration $\theta(t) = (\theta_1(t), \cdots, \theta_N(t))$,

$$r_N(K, t) := \Big| \frac{1}{N} \sum_{j=1}^N e^{i\theta_j(t)} \Big|.$$

Corollary 3.2.1. (Non-trivial Lower bound of $r_N(K, t)$). *Under the same assumptions as Lemma 3.1, we have*

$$r_N(K, t) \geq \cos \Big(\frac{\mathcal{D}(\theta^0)}{2} \Big), \quad t \geq 0.$$

Proof. We first observe that the Kuramoto real order parameter $r_N(K, t)$ is invariant under the rotational motion, i.e.,

$$\bar{r} = \frac{1}{N} \Big| \sum_{j=1}^N e^{i(\theta_j + \alpha)} \Big| = \frac{1}{N} \Big| e^{i\alpha} \sum_{j=1}^N e^{i\theta_j} \Big| = \frac{1}{N} \Big| \sum_{j=1}^N e^{i\theta_j} \Big| = r.$$

Hence without loss of generality, we can assume

$$|\theta_i(t)| \leq \frac{\mathcal{D}(\theta^0)}{2} \ (< \frac{\pi}{2}), \qquad t \geq 0.$$

Then by definition of Kuramoto's order parameter, we have

$$r_N(K, t) \ = \ \frac{1}{N} \Big| \sum_{j=1}^N e^{i\theta_j} \Big|$$

$$
\begin{aligned}
&= \frac{1}{N}\Big|\sum_{j=1}^{N}\cos\theta_j + i\sum_{j=1}^{N}\sin\theta_j\Big| \\
&\geq \frac{1}{N}\sum_{j=1}^{N}\cos\theta_j \\
&\geq \frac{1}{N}\sum_{j=1}^{N}\cos\Big(\frac{\mathcal{D}(\theta^0)}{2}\Big) = \cos\Big(\frac{\mathcal{D}(\theta^0)}{2}\Big).
\end{aligned}
$$

□

Lemma 3.2.2. *Suppose $\theta = \theta(t)$ is the set of N-values in $[0, 2\pi)$ satisfying*

$$|\theta_i - \theta_j| < \pi, \quad 1 \leq i, j \leq N.$$

Then we have the following estimates:

(i) For $l \neq M, m$,

$$\sin(\theta_M - \theta_m) + \sin(\theta_l - \theta_M) + \sin(\theta_m - \theta_l) \leq 0.$$

(ii) For $l \neq i, j$,

$$\sin(\theta_i - \theta_j) + \sin(\theta_l - \theta_i) - \sin(\theta_l - \theta_j) = C_{ij}^l \sin(\theta_i - \theta_j),$$

where C_{ij}^l is given by the following:

$$C_{ij}^l := 1 - \frac{\cos(\frac{\theta_l - \theta_i}{2} + \frac{\theta_l - \theta_j}{2})}{\cos(\frac{\theta_j - \theta_i}{2})}.$$

Proof. (i) We use elementary properties of trigonometric functions and

$$0 \leq \theta_M - \theta_l,\ \theta_l - \theta_m < \pi,$$

to see

$$
\begin{aligned}
&\sin(\theta_M - \theta_m) + \sin(\theta_l - \theta_M) + \sin(\theta_m - \theta_l) \\
&\quad = \sin(\theta_M - \theta_l + \theta_l - \theta_m) + \sin(\theta_l - \theta_M) + \sin(\theta_m - \theta_l) \\
&\quad = \sin(\theta_M - \theta_l)\cos(\theta_l - \theta_m) + \cos(\theta_M - \theta_l)\sin(\theta_l - \theta_m)
\end{aligned}
$$

$$
\begin{aligned}
&-\sin(\theta_M-\theta_l)-\sin(\theta_l-\theta_m)\\
&=\sin(\theta_M-\theta_l)\big(\cos(\theta_l-\theta_m)-1\big)+\sin(\theta_l-\theta_m)\big(\cos(\theta_M-\theta_l)-1\big)\\
&\le 0.
\end{aligned}
$$

(ii) By direct calculation, we have

$$
\begin{aligned}
&\sin(\theta_i-\theta_j)+\sin(\theta_l-\theta_i)-\sin(\theta_l-\theta_j)\\
&\qquad=\sin(\theta_i-\theta_j)+2\sin\left(\frac{\theta_j-\theta_i}{2}\right)\cos\left(\frac{2\theta_l-\theta_i-\theta_j}{2}\right)\\
&\qquad=\sin(\theta_i-\theta_j)\Big[1-\frac{2\sin(\frac{\theta_j-\theta_i}{2})\cos(\frac{2\theta_l-\theta_i-\theta_j}{2})}{\sin(\theta_j-\theta_i)}\Big]\\
&\qquad=\sin(\theta_i-\theta_j)\Big[1-\frac{\cos(\frac{\theta_l-\theta_i}{2}+\frac{\theta_l-\theta_j}{2})}{\cos(\frac{\theta_j-\theta_i}{2})}\Big]\\
&\qquad=C_{ij}^l\sin(\theta_i-\theta_j).
\end{aligned}
\tag{3.2.4}
$$

□

Lemma 3.2.3. *Let $\theta=\theta(t)$ be the global smooth solution to the system* (3.1.1)-(3.1.2) *satisfying*

$$0<\mathcal{D}(\theta^0)<\pi,\quad \mathcal{D}(\Omega)>0,\quad K>K_e.$$

Then we have the following nonlinear Gronwall's inequality for $\mathcal{D}(\theta(t))$:

$$\dot{\mathcal{D}}(\theta)\le\mathcal{D}(\Omega)-K\sin(\mathcal{D}(\theta)),\quad \textit{where}\quad \dot{\mathcal{D}}(\theta(t)):=\frac{d}{dt}\mathcal{D}(\theta(t)).$$

Proof. It follows from the assumptions and Lemma 3.2.1 that

$$\mathcal{D}(\theta(t))\le\mathcal{D}(\theta^0)<\pi.$$

Hence we can apply the estimates in Lemma 3.2.2 for $\theta(t)$.

We now use (3.1.1) to see

$$
\begin{aligned}
\dot\theta_i - \dot\theta_j &= \Omega_i - \Omega_j + \frac{K}{N}\sum_{l=1}^{N}\Big(\sin(\theta_l-\theta_i) - \sin(\theta_l-\theta_j)\Big)\\
&= \Omega_i - \Omega_j + \frac{K}{N}\Big[-2\sin(\theta_i-\theta_j) + \sum_{l\neq i,j}\Big(\sin(\theta_l-\theta_i) - \sin(\theta_l-\theta_j)\Big)\Big]\\
&= \Omega_i - \Omega_j - K\sin(\theta_i-\theta_j)\\
&+ \frac{K}{N}\sum_{l\neq i,j}\Big(\sin(\theta_i-\theta_j) + \sin(\theta_l-\theta_i) - \sin(\theta_l-\theta_j)\Big).
\end{aligned}
\tag{3.2.5}
$$

We choose i and j to be the extremal indices:

$$i = M, \quad j = m.$$

Then for such choices, we have

$$
\begin{aligned}
\dot\theta_M - \dot\theta_m &= \Omega_M - \Omega_m - K\sin(\theta_M-\theta_m)\\
&+ \frac{K}{N}\sum_{l\neq M,m}\Big(\sin(\theta_M-\theta_m) + \sin(\theta_l-\theta_M) + \sin(\theta_m-\theta_l)\Big).
\end{aligned}
$$

We use Lemma 3.2.2 and $K > K_e$ to get

$$\dot\theta_M - \dot\theta_m \le \mathcal{D}(\Omega) - K\sin(\theta_M-\theta_m). \tag{3.2.6}$$

□

3.2.3 Convergence toward phase-locked states

In the following proposition, we show that any phase configurations whose diameter is in the range $(\frac{\pi}{2}, \pi)$ will be shrink to a smaller set $\mathcal{S}(\theta^0)$ in a finite-time.

Proposition 3.2.1. (Finite-time entrance to $\mathcal{S}(\theta^0)$) *Let $\theta = \theta(t)$ be the global smooth solution to the system* (3.1.1)-(3.1.2) *satisfying*

$$0 < \mathcal{D}(\theta^0) < \pi, \quad \mathcal{D}(\Omega) > 0, \quad K > K_e.$$

Then there exists $t_0 > 0$ such that

$$\mathcal{D}(\theta(t)) \leq \mathcal{D}^{\infty} \quad \textit{for} \quad t \geq t_0.$$

Proof. Although this proposition is essential for the proof of Theorem 3.2.1, it is rather lengthy and technical. Hence for the smooth flow of reading, we postpone its proof in Appendix A. □

As a direct application of Proposition 3.2.1, we obtain the exponential decay of the frequency diameter.

The proof of Theorem 3.2.1: First, the proof of upper bound estimate follows from Proposition 3.2.1 and Theorem 3.1.1. For the lower bound estimate, we use $\cos\theta \leq 1$ to find

$$\mathcal{D}(\omega(t)) \geq \mathcal{D}(\omega(t_0)) \exp[-K(t-t_0)], \quad t \geq t_0.$$

Hence the decay rate lies in the interval $[K\cos\mathcal{D}^{\infty}, K]$ in a large-time regime.

Remark 3.2.2. *1. The estimate given in Theorem 3.2.1 implies the asymptotic formation of phase-locked states (see Remark 3.1.1(1)):*

$$|\hat{\omega}_i(t)| \leq \mathcal{D}(\omega(t)) \to 0, \quad \textit{as } t \to \infty.$$

2. For identical Kuramoto oscillators, the optimal decay rate in the frequency diameter is exactly K as in [31]. Hence for the non-identical case, the time-decay of the frequency diameter can be slower than that of identical oscillators.

3. The condition $\mathcal{D}(\theta^0) < \pi$ can be relaxed to $\mathcal{D}(\theta^0) \leq \pi$, when the initial configuration is not the type of bipolar configuration. In this case, the diameter of the initial configuration begins to monotone decrease at $t = 0+$ so that $\mathcal{D}(\theta(\varepsilon))$ for $0 < \varepsilon \ll 1$. Then we can apply our theorem to conclude the formation of phase-locked states.

Before we close this section, we present an estimate regarding the coupling strength K and the upper bound of asymptotic size of phase diameter. Basically the following theorem says that if we want to control the size of phase-diameter to make it smaller than $\varepsilon > 0$, then we need a coupling strength larger than $|\mathcal{O}(1)|\frac{1}{\varepsilon}$.

Proposition 3.2.2. *For any $0 < \varepsilon \ll 1$, let $\theta = \theta(t)$ be the global smooth solution to the system* (3.1.1)-(3.1.2) *with initial configuration satisfying*

$$0 < \mathcal{D}(\theta^0) < \pi.$$

If the coupling strength K satisfies

$$K > \max\left\{K_e, \frac{2\mathcal{D}^\infty \mathcal{D}(\Omega)}{\varepsilon \sin \mathcal{D}^\infty}\right\},$$

then there exists a positive time T_* such that

$$\mathcal{D}(\theta(t)) \le \varepsilon, \qquad t \ge T_*.$$

Proof. This proof is given in Appendix A. □

Remark 3.2.3. *We show that the diameter $\mathcal{D}(\theta(t))$ is inversely proportional to K, i.e., we can reduce the size of the diameter $\mathcal{D}(\theta(t))$ by increasing K, of course for large time t (explicitly, $t \ge T_* = \frac{1}{K} \log\left(\frac{2(\mathcal{D}(\theta(t_0)) - \mathcal{D}(\Omega)/\bar{K})}{\varepsilon}\right)$). Thus we obtain same results with the identical Kuramoto oscillator in case $K = \infty$.*

3.3 Quantitative estimates toward the phase-locked states

In the previous section, we have discussed the asymptotic formation of phase-locked states arising from some admissible initial configurations, which basically deal with the large-time dynamics of solutions. Hence in this section, we focus on more quantitative estimates in the intermediate-time zone which were reflected in the following questions:

- What is the total number of collisions during the evolution to the phase-locked states ?
- In the relaxation process toward the phase-locked states, how do the transversal phase-differences evolve ?

In the following two subsections, we will try to answer the above questions.

3.3.1 Finiteness on the number of collisions

In this part, we discuss collisions between oscillators during the complete-frequency synchronization process. Here collisions denote the zero congruence of phase differences modulo 2π. In [31], the number of collisions was discussed using the theory of real analytic functions. Note that the collision times in the i and j-th oscillators simply denote zeros of the real-analytic function $\theta_{ij} := \theta_i - \theta_j$. Hence the set of collision times must be countable and isolated due the theory of real-analytic functions. Moreover it can not have accumulation points. Therefore in any finite-time interval, the set of collision times must be finite. Otherwise, by the Bolzano-Weierstrass theorem, it must have accumulation points that is not true. However as the length of the interval goes to infinity, the aforementioned argument does not yield the finiteness of collision times. Hence a priori the occurrence of re-collisions is not clear.

Throughout the chapter, as long as there is no confusion, we denote t_0 to be the generic finite-time.

Let $\theta = \theta(t)$ be the global solution to the Kuramoto model (3.1.1)-(3.1.2). For $i \neq j$, we set

$$\theta_{ij} := \theta_i - \theta_j, \quad \Omega_{ij} := \Omega_i - \Omega_j.$$

In our main theorem, we will show that the oscillators with larger natural frequencies will eventually go ahead in a finite-time.

Theorem 3.3.1. (Estimate on eventual ordering by natural frequencies) *Let $\theta = \theta(t)$ be the global smooth solution to the system* (3.1.1)-(3.1.2) *satisfying*

$$0 < \mathcal{D}(\theta^0) < \pi, \quad K > K_e.$$

If $\Omega_i < \Omega_j$, then there exists a positive time t_{ij}^* such that

$$\theta_i(t) < \theta_j(t) \quad \textit{for} \quad t \geq t_{ij}^*.$$

That is the oscillators with large natural frequencies will be advanced to the front starting from any initial configurations. Actually, we can precisely count the collision times.

Lemma 3.3.1. *Let $\theta = \theta(t)$ be the global smooth solution to the system* (3.1.1)-(3.1.2) *satisfying*

$$0 < \mathcal{D}(\theta^0) < \pi, \quad \mathcal{D}(\Omega) > 0, \quad K > K_e.$$

(i) We assume for $i \neq j$,

(1) $\Omega_{ij} > 0$.
(2) *There exist t_0, T such that* $\theta_{ij}(t) > 0, \qquad t \in [t_0, T)$.

Then we have

$$\theta_{ij}(t) \geq \theta_{ij}(t_0)e^{-KU(N,\mathcal{D}(\theta^0))(t-t_0)} + \frac{\Omega_{ij}}{KU(N,\mathcal{D}(\theta^0))}\big(1 - e^{-KU(N,\mathcal{D}(\theta^0))(t-t_0)}\big), \qquad t \in [t_0, T),$$

where U is given by the following relation:

$$U(N, \mathcal{D}(\theta^0)) := 1 - \frac{N-2}{N}\Big(1 - \frac{1}{\cos\frac{\mathcal{D}(\theta^0)}{2}}\Big).$$

(ii) Suppose (i, j) satisfies

$$\theta_i(t_0) > \theta_j(t_0) \quad \textit{and} \quad \Omega_i > \Omega_j \quad \textit{for some} \quad t_0 \in \mathbb{R}_+$$

Then i and j-oscillators will not meet after $t = t_0$, i.e.,

$$\theta_i(t) > \theta_j(t), \quad t > t_0.$$

Proof. We leave this proof in Appendix C. □

Remark 3.3.1. *1. In fact, we can use $\mathcal{D}^\infty$ instead of $\mathcal{D}(\theta^0)$ in $U(N, \mathcal{D}(\theta^0))$ by Proposition 3.2.1.*

2. For a two-oscillator system, the mean-field coupling term C^l_{ij} vanishes. Hence we have

$$\dot{\theta} + K \sin\theta = \Omega,$$

where $\theta := \theta_1 - \theta_2$ and $\Omega := \Omega_1 - \Omega_2$.

3. Note that there are no re-collisions on the circle, since Lemma 3.2.1 shows that

$$|\theta_i(t) - \theta_j(t)| \leq \mathcal{D}(\theta(t)) \leq \mathcal{D}(\theta^0) < \pi.$$

The proof of Theorem 3.3.1: Depending on the initial phases θ_i^0 and θ_j^0, we consider three cases:

$$\text{Either } \theta_i^0 < \theta_j^0, \quad \theta_i^0 > \theta_j^0 \quad \text{or} \quad \theta_i^0 = \theta_j^0.$$

Case 1 ($\theta_i^0 = \theta_j^0$): If $\theta_i^0 = \theta_j^0$, then it is easy to see that

$$(\dot{\theta}_j - \dot{\theta}_i)(0+) = \Omega_j - \Omega_i > 0.$$

Hence we have

$$\theta_j(0+) > \theta_i(0+),$$

which reduces to Case 3 below.

Case 2 ($\theta_i^0 < \theta_j^0$): In this case, by Lemma 3.3.1(ii), we have

$$\theta_i(t) < \theta_j(t), \quad t \geq 0.$$

Hence we can choose $t_{ij}^* := 0$.

Case 3 ($\theta_i^0 > \theta_j^0$): In this case, we will show that there is a finite-time t_0 such that

$$\theta_i(t_0) \leq \theta_j(t_0).$$

Suppose not, i.e.,

$$\theta_{ij}(t) := \theta_i(t) - \theta_j(t) > 0, \quad t \geq t_0.$$

Then by Proposition 3.2.1, there exists a finite time $t_*(\geq t_0)$ such that

$$\mathcal{D}(\theta(t)) \leq \mathcal{D}^\infty \in (0, \frac{\pi}{2}), \quad t \geq t_*.$$

Since $\cos \mathcal{D}^\infty > 0$ and $|\theta_i(t) - \theta_j(t)| \leq \mathcal{D}^\infty, \ t \geq t_*$, we have

$$\cos \mathcal{D}^\infty \leq \frac{\cos(\frac{\theta_l - \theta_i}{2} + \frac{\theta_l - \theta_j}{2})}{\cos \frac{\theta_i - \theta_j}{2}} \leq \frac{1}{\cos \frac{\mathcal{D}^\infty}{2}}, \quad t \geq t_*.$$

Note that we also have

$$L(N, \mathcal{D}^\infty) := 1 - \frac{N-2}{N}(1 - \cos \mathcal{D}^\infty) \leq \beta_{ij}(N, \theta), \quad t \geq t_*.$$

Then we use the relation (7.2.1) and

$$\frac{KL(N,\mathcal{D}^\infty)\sin\mathcal{D}^\infty}{\mathcal{D}^\infty}\theta_{ij} \le KL(N,\mathcal{D}^\infty)\sin\theta_{ij} \le K\beta_{ij}(N,\theta)\sin\theta_{ij}$$

to get Gronwall's inequality for θ_{ij}:

$$\dot{\theta}_{ij} + \frac{KL\sin\mathcal{D}^\infty}{\mathcal{D}^\infty}\theta_{ij} \le \Omega_{ij}.$$

This yields

$$\begin{aligned}\theta_{ij}(t) &\le \theta_{ij}(t_*)e^{-KL\frac{\sin\mathcal{D}^\infty}{\mathcal{D}^\infty}(t-t_*)} + \frac{\mathcal{D}^\infty\Omega_{ij}}{KL\sin\mathcal{D}^\infty}(1-e^{-KL\frac{\sin\mathcal{D}^\infty}{\mathcal{D}^\infty}(t-t_*)}) \\ &= \frac{\mathcal{D}^\infty\Omega_{ij}}{KL\sin\mathcal{D}^\infty} + \Big(\theta_{ij}(t_*) - \frac{\mathcal{D}^\infty\Omega_{ij}}{KL\sin\mathcal{D}^\infty}\Big)e^{-KL\frac{\sin\mathcal{D}^\infty}{\mathcal{D}^\infty}(t-t_*)}.\end{aligned}$$

On the other hand, note that

$$\frac{\mathcal{D}^\infty\Omega_{ij}}{KL\sin\mathcal{D}^\infty} < 0, \qquad \lim_{t\to\infty}\Big(\theta_{ij}(t_*) - \frac{\mathcal{D}^\infty\Omega_{ij}}{KL\sin\mathcal{D}^\infty}\Big)e^{-KL\frac{\sin\mathcal{D}^\infty}{\mathcal{D}^\infty}(t-t_*)} = 0.$$

Hence there exists a finite time t_1 such that

$$\theta_{ij}(t_1) < \frac{1}{2}\frac{\mathcal{D}^\infty\Omega_{ij}}{KL\sin\mathcal{D}^\infty} < 0.$$

This gives the contradiction. Hence there is a finite-time crossing such that

$$\theta_i(t_0) \le \theta_j(t_0).$$

Therefore by Lemma 3.3.1(ii), we have

$$\theta_i(t) < \theta_j(t), \qquad t \ge t_0.$$

Remark 3.3.2. *1. The result in Theorem 3.3.1 asserts the collision times are less than or equal to $N(N-1)/2$, since we can consider t_0 to be a time after collision time between different two oscillators and thus these oscillators do not collide for $t \ge t_0$. This yields there is no collision after finite time. For simplicity, if all natural frequencies are different, then the collision number is precisely the number of permutations of N-cycle $\{\Omega_1,\ldots,\Omega_N\}$.*

2. For identical-oscillators ($\Omega_{ij} = 0, i \neq j$), by same argument as in the proof of Lemma 3.3.1(ii), we have

$$\theta_{ij}(t) \geq \theta_{ij}(0)e^{-KU(N,\mathcal{D}(\theta^0))t}, \quad t \in [0, \infty).$$

On the other hand, by Theorem 3.3.1, we obtain

$$\theta_{ij}(t) \leq \theta_{ij}(t_*)e^{-KL\frac{\sin \mathcal{D}^\infty}{\mathcal{D}^\infty}(t-t_*)}, \quad t \geq t_*.$$

Hence we know that there is no finite-time phase collisions between two identical oscillators with different initial phases, i.e., if $\theta_i^0 \neq \theta_j^0$, then the i and j-th identical oscillators never collide in a finite-time. We also know that if $\Omega_{ij} = 0$, then

$$\theta_{ij}(t) \to 0 \quad as \quad t \to \infty,$$

from the second inequality of the above.

3.3.2 Estimate on the transversal phase differences

In this part, we study the lower and upper bounds for the transversal phase differences during the complete-frequency synchronization process. For this, we explicitly provide the lower and upper bounds for the transversal phase differences $\theta_{ij} = \theta_i - \theta_j,\ i \neq j$.

We first state our main theorem of this subsection. It is obtained by combining Proposition 3.3.1 and Proposition 3.3.2.

Theorem 3.3.2. *Let $\theta = \theta(t)$ be the global smooth solution to the system* (3.1.1)-(3.1.2) *satisfying*

$$0 < \mathcal{D}(\theta^0) < \pi \quad and \quad K > K_e.$$

Then we have

$$\sin^{-1}\left(\frac{\Omega_{ij}}{KU}\right) \leq \lim_{t\to\infty} \theta_{ij}(t) \leq \mathcal{D}^\infty.$$

Moreover, if

$$\left|\frac{\Omega_{ij}}{KL}\right| \leq 1 \quad and \quad \sin^{-1}\left(\frac{\Omega_{ij}}{KL}\right) \leq \mathcal{D}^\infty,$$

then we obtain

$$\sin^{-1}\Big(\frac{\Omega_{ij}}{KU}\Big) \le \lim_{t\to\infty}\theta_{ij}(t) \le \sin^{-1}\Big(\frac{\Omega_{ij}}{KL}\Big).$$

Remark 3.3.3. *In the thermodynamics limit ($N \to \infty$), we have*

$$U(N,\mathcal{D}^\infty) \to \frac{1}{\cos\frac{\mathcal{D}^\infty}{2}} \quad \textit{and} \quad L(N,\mathcal{D}^\infty) \to \cos\mathcal{D}^\infty.$$

In the sequel, thanks to Theorem 3.3.1, without loss of generality, we may assume that

$$0 < \mathcal{D}(\theta^0) < \pi, \qquad K > K_e, \qquad \theta_i > \theta_j, \qquad \Omega_i > \Omega_j.$$

We now set $\theta_{ij}^* \in (0, \frac{\pi}{2})$ to be the solution of the following trigonometric equation:

$$\sin x = \frac{\Omega_{ij}}{KU}. \tag{3.3.1}$$

Since $K > K_e$ and $U > 1$, we have

$$\sin\theta_{ij}^* = \frac{\Omega_{ij}}{KU} \le \frac{\mathcal{D}(\Omega)}{KU} < \frac{\mathcal{D}(\Omega)}{K} < \sin\mathcal{D}(\theta^0).$$

It follows from (7.2.1) and (7.2.2) that

$$\dot\theta_{ij} \ge \Omega_{ij} - KU\theta_{ij}.$$

We next set

$$\lambda_{ij} := \sqrt{K^2U^2 - \Omega_{ij}^2} > 0 \quad \text{and} \quad A_{ij}^{\pm} := \Omega_{ij}\tan\Big(\frac{\theta_{ij}}{2}\Big) - KU \pm \lambda_{ij}, \quad i,j = 1,\cdots,N.$$

Lemma 3.3.2. *(i) Let $\theta = \theta(t)$ be the global smooth solution to the equation* (3.1.1)-(3.1.2)*. Suppose there exists t_0 such that*

$$\theta_{ij}^* < \theta_{ij}(t_0).$$

Then there exists a decreasing function g(t) with the following properties:

$$g(t) < \theta_{ij}(t), \qquad \lim_{t\to\infty} g(t) = \theta_{ij}^*,$$

and we have

$$\theta_{ij}^* < \theta_{ij}(t), \quad t \geq t_0.$$

(ii) Let $\theta = \theta(t)$ *be the global smooth solution to the system* (3.1.1)-(3.1.2) *satisfying the following assumption: for some* $t_0 \in \mathbb{R}_+$,

$$\sin\theta_{ij}(t) < \sin\theta_{ij}^* \quad \textit{for} \quad t \geq t_0.$$

Then we have

$$A_{ij}^- < A_{ij}^+ < 0.$$

where θ_{ij}^* *is defined by the equation* (3.3.1).

Proof. For the proof, we leave it to Appendix C. □

Proposition 3.3.1. (Lower bound estimate) *Let* $\theta = \theta(t)$ *be the global smooth solution to the system* (3.1.1)-(3.1.2) *satisfying the following conditions*

$$0 < \mathcal{D}(\theta^0) < \pi, \qquad K > K_e.$$

Suppose there exists $t_0 > 0$ *such that*

$$\sin\theta_{ij}(t) < \sin\theta_{ij}^* \quad \textit{for} \quad t \geq t_0.$$

Then there exists $t_* > 0$ *such that*

$$\begin{aligned} &(i)\ \Omega_{ij}\tan\frac{\theta_{ij}(t)}{2} > \frac{KU - \lambda_{ij} - (KU + \lambda_{ij})e^{C_1 - \lambda_{ij}t}}{1 - e^{C_1 - \lambda_{ij}t}}, \quad t \geq t_*, \\ &(ii)\ \lim_{t\to\infty}\theta_{ij}(t) = \theta_{ij}^*, \end{aligned}$$

where

$$C_1 := \log\left|\frac{\Omega_{ij}\tan(\frac{\theta_{ij}(t_0)}{2}) - KU - \lambda_{ij}}{\Omega_{ij}\tan(\frac{\theta_{ij}(t_0)}{2}) - KU + \lambda_{ij}}\right|.$$

Proof. For the proof, we leave it to Appendix C. □

Proposition 3.3.2. (Upper bound estimate) *Let* $\theta = \theta(t)$ *be the global smooth solution to the system* (3.1.1)-(3.1.2) *satisfying the following conditions*

$$0 < \mathcal{D}(\theta^0) < \pi, \qquad K > K_e.$$

Suppose $\theta_{ij}(t_0) > \sin^{-1}(\frac{\Omega_{ij}}{KU})$ *for some* $t_0 \geq 0$. *Then we have*

$$\sin^{-1}\left(\frac{\Omega_{ij}}{KU}\right) < \theta_{ij}(t) < \sin^{-1}\left(\frac{\Omega_{ij}}{KL}\right) \quad \text{for} \quad t \geq t_0,$$

if

$$\sin^{-1}\left(\frac{\Omega_{ij}}{KL}\right) \leq \mathcal{D}^\infty \quad \text{and} \quad \left|\frac{\Omega_{ij}}{KL}\right| < 1.$$

Otherwise we obtain

$$\sin^{-1}\left(\frac{\Omega_{ij}}{KU}\right) < \theta_{ij}(t) < \mathcal{D}^\infty \quad \text{for} \quad t \geq t_0.$$

Proof. It follows from Proposition 3.2.1 and Lemma 3.3.2 that

$$\sin^{-1}\left(\frac{\Omega_{ij}}{KU}\right) < \theta_{ij}(t) < \mathcal{D}^\infty \quad \text{for} \quad t \geq t_0.$$

If $\sin^{-1}(\frac{\Omega_{ij}}{KL}) \leq \mathcal{D}^\infty$ and $|\frac{\Omega_{ij}}{KL}| < 1$, then by same argument as in Proposition 3.3.1 we have

$$\sin^{-1}\left(\frac{\Omega_{ij}}{KU}\right) < \theta_{ij}(t) < \sin^{-1}\left(\frac{\Omega_{ij}}{KL}\right) \quad \text{for} \quad t \geq t_0.$$

□

3.4 Numerical Simulations

In this section, we provide several numerical simulations and compare them with aforementioned analytical results in Section 3.2 and Section 3.3.

For numerical simulations, we employed the fourth order Runge-Kutta scheme.

3.4.1 Formation of phase-locked states

We first explain the numerical simulation performed in Figure 3.1.(a). For the simulation, we employed

$$N = 100, \qquad K = 4,$$

and the natural frequency Ω_i are randomly chosen from the uniform distribution on $[-1, 1]$ so that

$$\mathcal{D}(\Omega) = 1.9426.$$

The initial phase configuration $\{\theta_i^0\}_{i=1}^{100}$ has been generated randomly from the uniform distribution on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and it satisfies

$$\frac{\pi}{2} < \mathcal{D}(\theta^0) = 2.5667 < \pi.$$

Note that the lower coupling bound K_e:

$$K_e = \frac{\mathcal{D}(\Omega)}{\sin \mathcal{D}(\theta^0)} = 3.5725.$$

Our choice for $K = 4$ satisfy the assumptions in Proposition 3.2.1.

It is easy to see from Figure 3.1.(a) that the phase diameter first shrink to the value less than $\mathcal{D}^\infty = 0.5749$ in a finite time. In fact the limit of $\mathcal{D}(\theta(t))$ is around the value 0.4960.

In Figure 3.1.(b), the initial phase configuration $\{\theta_i^0\}_{i=1}^{5}$ has been selected randomly from the uniform distribution on $[-\frac{\pi}{3}, \frac{\pi}{3}]$ and it satisfies the sufficient condition in Theorem 3.3.1:

$$\mathcal{D}(\theta^0) = 0.9652 < \pi.$$

The natural frequency Ω_i are randomly chosen from the uniform distribution on $[-1, 1]$ so that

$$\mathcal{D}(\Omega) = 0.7777.$$

In the simulation we choose $K = 1 > 0.9459 = K_e$. In Figure 1.(b), we can see that the asymptotic phase-locked state is arranged in decreasing order of natural frequency which is consistent with Theorem 3.3.1.

3.4.2 Estimate on the number of collisions

In this part, we present a numerical simulation result for Theorem 3.3.1

In Figure 3.2.(a),(b), the number of collisions for non-identical Kuramoto oscillators is numerically computed.

θ^0	$\ell(0.5)$	$\ell(1)$	$\ell(2)$	$\ell(3)$
(a) $\mathcal{D}(\theta^0) = 0.7674$	2014	2077	2082	2082
(b) $\mathcal{D}(\theta^0) = 3$	0	3805	4950	4950

Here $\ell(t)$:= number of accumulated collisions up to time t.

In Figure 3.2(a), we choose the initial phase configuration from the uniform distribution on $[0, \pi]$ such that $\mathcal{D}(\theta^0) = 0.7674$, and natural frequencies are obtained randomly from the uniform distribution $[0, 10]$ such that $\mathcal{D}(\Omega) = 4.7937$. In this case, we have

$$K_e = 6.9046.$$

For the simulation we have chosen $K = 7 > 6.9046$.

In Figure 3.2.(b), we take the initial phase configuration and natural frequencies as follows:

$$\theta_i^0 := 1.5 - \frac{3(i-1)}{99} \quad \text{and} \quad \Omega_i := -0.24 + \frac{0.48(i-1)}{99} \quad \text{for} \quad i \in \{1, \ldots, 100\}.$$

and in this case , we obtain

$$K_e = 3.4014 \quad \text{and} \quad K = 3.5.$$

In this setting, we expect the number of collisions is

$$\frac{100(100-1)}{2} = 4950,$$

which exactly coincide with numerical simulations(see the above table).

3.4.3 Estimate on the evolution of phase-differences

In Figure 3.2.(c), we choose the initial configurations the same as in Figure 3.1.(a). On the other hand, note that

$$\mathcal{D}^\infty = \sin^{-1}(\sin \mathcal{D}(\theta^0)) = 0.5749, \quad K_e = \frac{\mathcal{D}(\Omega)}{\sin(\mathcal{D}(\theta^0))} = 3.5725.$$

In order to have K which is bigger than K_e, we set

$$K = 4.$$

In this setting, we also have

$$L = 0.8424, \quad U = 1.0419 \quad \text{and} \quad \lim_{t\to\infty} \theta_{ij}(t) = 0.1837.$$

Therefore,

$$\sin^{-1}(\frac{\Omega_{ij}}{KU}) = 0.1731 \le \lim_{t\to\infty} \theta_{ij}(t) = 0.1837 \le \sin^{-1}(\frac{\Omega_{ij}}{KL}) = 0.2146.$$

This is consistent with Theorem 3.3.2.

3.4.4 Transition and relaxation stages

In this subsection, we numerically investigate the dynamic features between transition stage and relaxation stage. For the illustration of two dynamic stages, we choose two initial data whose diameters are larger or smaller than $\frac{\pi}{2}$.

In Figure 3.3.(a), the initial phase configuration $\{\theta_i^0\}_{i=1}^{100}$ and the natural frequency $\{\Omega_i\}_{i=1}^{100}$ have been generated randomly from the uniform distributions on $\Big[-\frac{7\pi}{16}, \frac{7\pi}{16}\Big]$ and $[-1, 1]$, respectively, and they satisfy

$$\frac{\pi}{2} < \mathcal{D}(\theta^0) = 2.6298 < \pi, \quad \text{and} \quad \mathcal{D}(\Omega) = 1.9871.$$

In this setting, we get

$$K_e = 4.0576,$$

and we choose the coupling strength $K = 4.1$.

In Figure 3.3.(a), the lines denote the numerical results in log scale respectively. We can see from Figure 3.3.(a) that in a short time interval, the decay of the frequency diameter is slower than that of later time interval, and then it eventually decays exponentially. This slower decay phenomenon occurs in the transition stage.

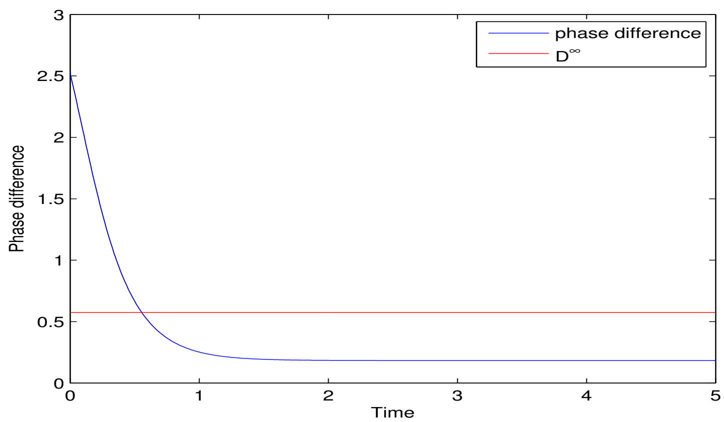
In Figure 3.3.(b), we randomly selected the initial phases $\{\theta_i^0\}_{i=1}^{100}$ and the natural frequencies $\{\Omega_i\}_{i=1}^{100}$ from the uniform distribution on $[-\frac{\pi}{5}, \frac{\pi}{5}]$ and $[-1, 1]$, respectively:

$$\mathcal{D}(\theta^0) = 1.1899 < \frac{\pi}{2}, \quad \text{and} \quad \mathcal{D}(\Omega) = 1.9831.$$

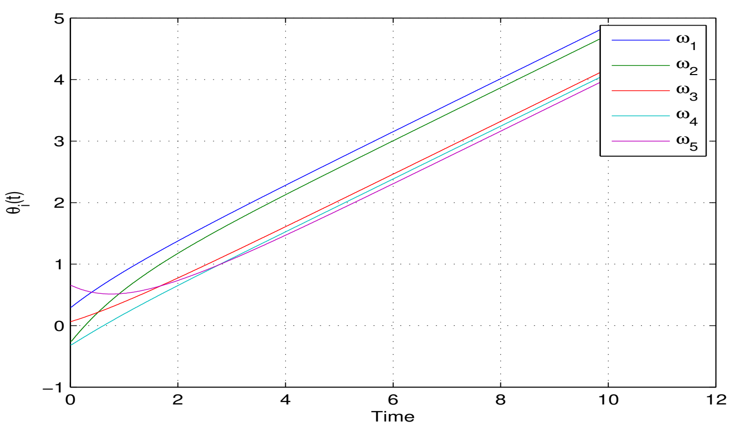
In this case, the initial configuration is already in the relaxation stage. By the definition of K_e, we can take the K_e and K as follows:

$$K_e = 2.1362, \quad K = 2.2362.$$

In this figure, the lines denote numerical results in log scale. We can see from Figure 3.3.(b) that the slope of the diameter of frequencies seems to be a straight line from the beginning. Of course, we cannot draw any general statement for the transition and relaxation stages from finite number of numerical simulations. We leave this interesting issues for future work.

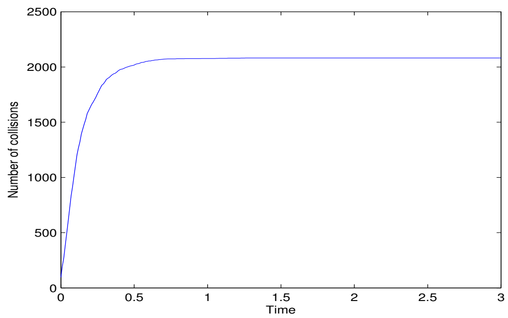


(a) $\mathcal{D}(\theta(t))$ and $\mathcal{D}^{\infty}$

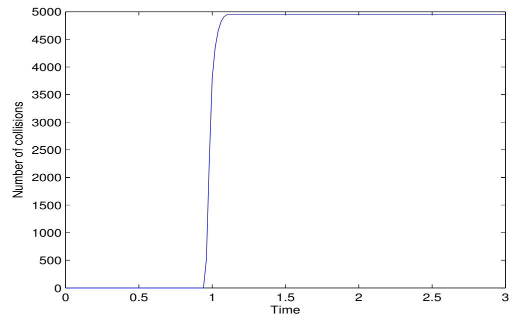


(b) $\theta_i(t)$

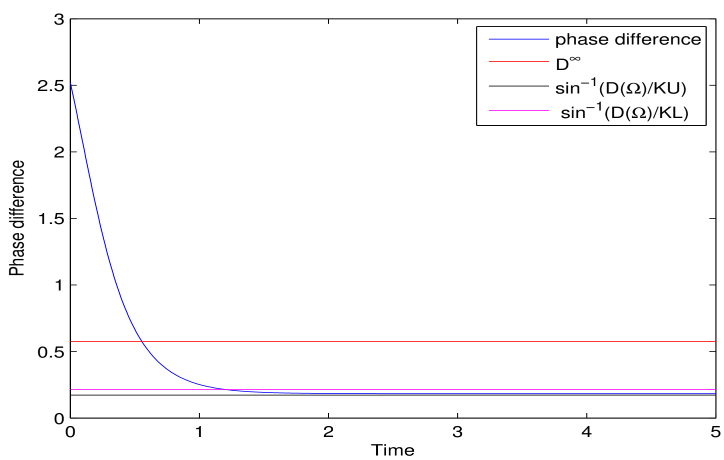
Figure 3.1: (a) The relation between $\mathcal{D}(\theta(t))$ and $\mathcal{D}^{\infty}$. Lines denote numerical simulation's results. The result show that the limit of diameter is smaller than $\mathcal{D}^{\infty}$. (b) Dynamics of phases of five non-identical oscillators under the initial phase configuration with $\mathcal{D}(\theta^0) = 0.9652$. Lines denote the result of numerical simulations. Simulation results show that the asymptotic phase-locked state are arranged according to the size of natural frequencies, where $\{\omega_i\}_{i=1}^5$ are natural frequencies of $\{\theta_i^0\}_{i=1}^5$ respectively satisfying $\omega_1 > \omega_2 > \omega_3 > \omega_4 > \omega_5$.



(a)

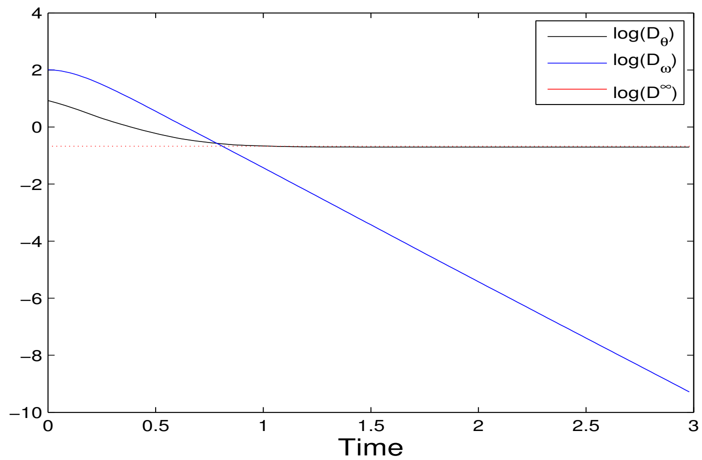


(b)

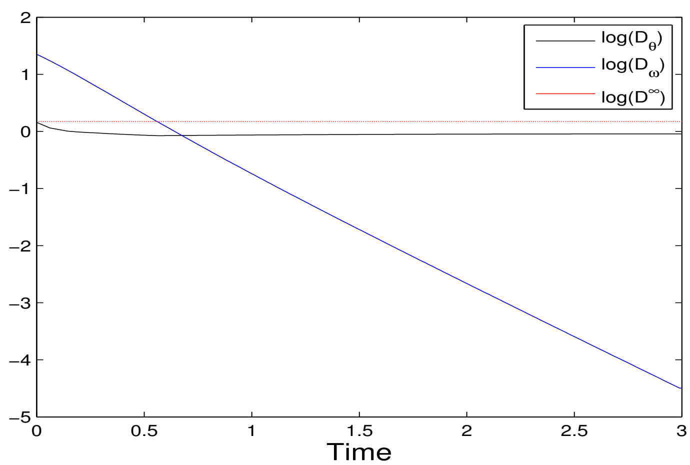


(c) boundedness for the phase difference

Figure 3.2: (a),(b) We can see that there is no collision after a finite-time and number of collisions is less than or equal to $\frac{100(99)}{2} = 4950$ which is the possible maximal number of collisions. (a) Initial configuration $\mathcal{D}(\theta^0) = 0.7674$ and $\mathcal{D}(\Omega) = 4.7937$. (b) Initial configuration $\mathcal{D}(\theta^0) = 3$ and $\mathcal{D}(\Omega) = 0.48$. (c) Here we consider the case $\sin^{-1}(\frac{\Omega_{ij}}{KL}) < \mathcal{D}^\infty$. We can see that the limit of phase difference is between $\sin^{-1}(\frac{\Omega_{ij}}{KU})$ and $\sin^{-1}(\frac{\Omega_{ij}}{KL})$.



(a) Comparison of two dynamic stages: $\mathcal{D}(\theta^0) > \frac{\pi}{2}$



(b) Comparison of two dynamic stages: $\mathcal{D}(\theta^0) \leq \frac{\pi}{2}$

Figure 3.3: (a) The boundary of two dynamic stages is represented by the red line, so we can see that configuration stays in the transition stages until about $t = 1$ which is the intersection of the red and black. In the transition stage, we can see that the frequency diameter is located over the straight line. (b) This case has no transition stage, and from the beginning, the frequency diameter seems decay with an exponential decay rate.

Chapter 4

Kuramoto oscillators with inertia

The purpose of this chapter is to extend the study of fast-slow dynamical systems theory for synchronization models, which started with [35, 43].

4.1 Derivation of fast-slow dynamics

In this section, we present a fast-slow reformulation for the system (2.2.2) in the limiting process $\varepsilon \to 0$.
Consider the following system of second-order ODEs:

$$m\ddot{\theta}_i + \dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N} a_j \sin(\theta_j - \theta_i), \quad i = 1, \cdots, N, \ t > 0, \tag{4.1.1}$$

subject to the initial data:

$$(\theta_i, \dot{\theta}_i)\Big|_{t=0} = (\theta_{i0}, \omega_{i0}), \tag{4.1.2}$$

where $a_j \in \{0, 1\}$ is the interaction weight indicating the impact of the j-th oscillator. The second-order system (4.1.1) can be rewritten as the first-order system:

$$
\begin{aligned}
\dot{\theta}_i &= \omega_i, \quad i = 1, \cdots, N, \ t > 0, \qquad (\theta_i, \omega_i) \in \mathbb{R}^2, \\
\dot{\omega}_i &= \frac{1}{m}\Big(-\omega_i + \Omega_i + \frac{K}{N}\sum_{j=1}^{N} a_j \sin(\theta_j - \theta_i)\Big).
\end{aligned}
$$

Next, we introduce the weighted Kuramoto order parameter $(r, \phi) \in \mathbb{R}_+ \times \mathbb{R}$:

$$
re^{i\phi} := \frac{1}{N}\sum_{j=1}^{N} a_j e^{i\theta_j}. \tag{4.1.3}
$$

Note that r is always bounded, i.e., $0 \le r \le 1$. We next divide (4.1.3) by $e^{i\theta_i}$ to get the equation:

$$
re^{i(\phi-\theta_i)} = \frac{1}{N}\sum_{j=1}^{N} a_j e^{i(\theta_j-\theta_i)},
$$

and we compare the real and imaginary parts of the above relation to find

$$
\begin{aligned}
r\cos(\phi - \theta_i) &= \frac{1}{N}\sum_{j=1}^{N} a_j \cos(\theta_j - \theta_i), \\
r\sin(\phi - \theta_i) &= \frac{1}{N}\sum_{j=1}^{N} a_j \sin(\theta_j - \theta_i).
\end{aligned}
$$

We differentiate the equation (4.1.3) with respect to t to get

$$
\dot{r}e^{i\phi} + ire^{i\phi}\dot{\phi} = \frac{i}{N}\sum_{j=1}^{N} a_j e^{i\theta_j}\dot{\theta}_j.
$$

We divide the resulting equation by $e^{i\phi}$ to find

$$
\dot{r} + ir\dot{\phi} = -\frac{1}{N}\sum_{j=1}^{N} a_j \sin(\theta_j - \phi)\dot{\theta}_j + \frac{i}{N}\sum_{j=1}^{N} a_j \cos(\theta_j - \phi)\dot{\theta}_j. \tag{4.1.4}
$$

We now take the real and imaginary parts of (4.1.4) to obtain

$$\dot{r} = -\frac{1}{N}\sum_{j=1}^{N} a_j \sin(\theta_j - \phi)\dot{\theta}_j,$$
$$\dot{\phi} = \frac{1}{rN}\sum_{j=1}^{N} a_j \cos(\theta_j - \phi)\dot{\theta}_j.$$

Thus, we obtain a coupled system for $(\theta_i, \omega_i, r, \phi)$:

$$\dot{\theta}_i = \omega_i, \quad i = 1, \cdots, N, \ t > 0,$$
$$\dot{\omega}_i = \frac{1}{m}\Big(- \omega_i + \Omega_i - Kr\sin(\theta_i - \phi)\Big),$$

and

$$\dot{r} = -\frac{1}{N}\sum_{j=1}^{N} a_j \sin(\theta_j - \phi)\omega_j,$$
$$\dot{\phi} = \frac{1}{rN}\sum_{j=1}^{N} a_j \cos(\theta_j - \phi)\omega_j.$$

We want to study the long-time dynamics and mean-field limit simultaneously, so we introduce the stretched time variable $t = \frac{\tau}{\varepsilon}$ where $0 \leq \tau \leq 1$. Thus, the system becomes

$$\frac{d\theta_i}{d\tau} = \frac{\omega_i}{\varepsilon}, \quad i = 1, \cdots, N, \ \tau > 0,$$
$$\frac{\omega_i}{d\tau} = \frac{1}{m\varepsilon}\Big(- \omega_i + \Omega_i - Kr\sin(\theta_i - \phi)\Big),$$
$$\frac{dr}{d\tau} = -\frac{1}{\varepsilon N}\sum_{j=1}^{N} a_j \sin(\theta_j - \phi)\omega_j,$$
$$\frac{d\phi}{d\tau} = \frac{1}{r\varepsilon N}\sum_{j=1}^{N} a_j \cos(\theta_j - \phi)\omega_j.$$

We then perform the following steps. We set

$$\varepsilon N = 1,$$

so that in principle as $\varepsilon \to 0$, we will get an infinite set of equations. However to prevent this event and maintain our system as finite dimensional, we set a truncation on a_j:

$$a_j = \begin{cases} 1, & j \leq M, \\ 0, & j > M. \end{cases}$$

Thus, we have a coupled fast-slow system:

$$\begin{aligned} \frac{d\theta_i}{d\tau} &= \frac{\omega_i}{\varepsilon}, \quad i = 1, \cdots, M, \ \tau > 0, \\ \frac{d\omega_i}{d\tau} &= \frac{1}{m\varepsilon}\Big(-\omega_i + \Omega_i - Kr\sin(\theta_i - \phi)\Big), \\ \frac{dr}{d\tau} &= -\sum_{j=1}^{M} \sin(\theta_j - \phi)\omega_j, \\ \frac{d\phi}{d\tau} &= \frac{1}{r}\sum_{j=1}^{M} \cos(\theta_j - \phi)\omega_j. \end{aligned} \tag{4.1.5}$$

Note that the system (4.1.5) is the sum of the fast (4.1.6) and slow (4.1.7) parts respectively: For $i \in \{1, \cdots, M\}$,

$$\begin{aligned} &\frac{d\theta_i}{dt} = \omega_i, \quad \frac{d\omega_i}{dt} = \frac{1}{m}\Big(-\omega_i + \Omega_i - Kr\sin(\theta_i - \phi)\Big), \quad t > 0, \\ &\frac{dr}{dt} = 0, \quad \frac{d\phi}{dt} = 0, \end{aligned} \tag{4.1.6}$$

and

$$\begin{aligned} &\frac{d\theta_i}{d\tau} = 0, \quad \frac{d\omega_i}{d\tau} = 0, \quad \tau > 0, \\ &\frac{dr}{d\tau} = -\sum_{j=1}^{M} \sin(\theta_j - \phi)\omega_j, \quad \frac{d\phi}{d\tau} = \frac{1}{r}\sum_{j=1}^{M} \cos(\theta_j - \phi)\omega_j. \end{aligned} \tag{4.1.7}$$

Since θ_j only enters the right hand side of (4.1.6) - (4.1.7) via sine and cosine functions, without loss of generality we may restrict θ_j to the interval $[-\pi, \pi]$ and identify all θ_i mod 2π as the same θ_i.

4.2 Invariant measure for the fast system

In this section, we study the invariant measure of the fast system, which has a support in the ω-limit set of the fast system.

Consider an autonomous two-dimensional system in $\mathbb{R}^2$:

$$\begin{aligned} \frac{dx}{dt} &= y, \quad t > 0, \\ \frac{dy}{dt} &= \frac{1}{m}\Big(-y + \Omega - Kr_0 \sin(x - \phi_0)\Big), \end{aligned} \tag{4.2.1}$$

subject to initial data:

$$(x, y)(0) = (x_0, y_0), \tag{4.2.2}$$

where $r_0 \in (0, 1], \phi_0 \in \mathbb{R}$ and $\Omega \geq 0$ are constants.

Note that the system (4.2.1) is periodic in x and (θ_i, ω_i) for the fast system (4.1.6) satisfies (4.2.1) with $\Omega = \Omega_i$ for $i = 1, \cdots, M$, while the equilibrium solutions to (4.2.1) should satisfy the relation:

$$y = 0, \qquad \Omega = Kr_0 \sin(x - \phi_0). \tag{4.2.3}$$

This means that if the system (4.2.1) has equilibrium solutions, then they are the equilibrium solutions to the system (4.2.1) without inertia, i.e., $m = 0$. Thus, the presence of inertia does not affect the structure of the equilibrium solutions, although it is known in [14] that inertia can modulate the relaxation speed toward the phase-locked states depending on the relative size of m and K.

Our fast system (4.2.1) is analogous to the equation for a damped pendulum with an applied constant torque (in our case $\frac{\Omega}{m}$) and the qualitative behavior of its solution has already been addressed in previous studies (e.g. [48, 61]). In particular, Levi et al [48] made an extensive classification of the dynamic behavior of solutions depending on the relative sizes of Ω and Kr_0. Below, we list the results of Levi et al which has been slightly modified in our setting without proofs.

4.2.1 Subcritical regime ($Kr_0 > \Omega$)

In these regimes, x satisfies the same trigonometric equation as that of the Kuramoto model without inertia. Thus it follows from [13] that the explicit formula can be given as follows:

$$\begin{aligned} t\sqrt{(Kr_0)^2 - \Omega^2} &= \log \Big| \frac{\Omega \tan \frac{x(t)-\phi_0}{2} - Kr_0 - \sqrt{(Kr_0)^2 - \Omega^2}}{\Omega \tan \frac{x(t)-\phi_0}{2} - Kr_0 + \sqrt{(Kr_0)^2 - \Omega^2}} \Big| \\ &- \log \Big| \frac{\Omega \tan \frac{x_0-\phi_0}{2} - Kr_0 - \sqrt{(Kr_0)^2 - \Omega^2}}{\Omega \tan \frac{x_0-\phi_0}{2} - Kr_0 + \sqrt{(Kr_0)^2 - \Omega^2}} \Big|. \end{aligned}$$

However, the above explicit formula is not useful for finding the invariant measure for the fast system. The equilibria can be found explicitly as follows: for $n = 0, \pm 1, \pm 2, \cdots$,

$$\begin{aligned} (x^e_{1n}, y^e_{1n}) &= (\phi_0 + \sin^{-1}\Big(\frac{\Omega}{Kr_0}\Big) + 2n\pi, 0), \quad \text{(Stable node)}, \\ (x^e_{2n}, y^e_{2n}) &= (\phi_0 + \pi - \sin^{-1}\Big(\frac{\Omega}{Kr_0}\Big) + 2n\pi, 0), \quad \text{(unstable saddle)}. \end{aligned}$$

The large-time behavior of the general solution can be seen in the following proposition.

Proposition 4.2.1. [48] *Suppose that the Ω and K satisfy*

$$0 \le \frac{\Omega}{Kr_0} < 1.$$

Then there exists a positive number $\sigma_* = \sigma_*\Big(\frac{\Omega}{Kr_0}\Big) > 0$ such that

1. *For $0 < \frac{1}{Kr_0} < \sigma_*$, there exists an exponentially stable running periodic orbit.*
2. *For $\frac{1}{Kr_0} > \sigma_*$, every orbit tends to one of the equilibria.*
3. *For $\frac{1}{Kr_0} = \sigma_*$, the state space is split into two regions. i.e., all orbits in the upper region tend to the boundary between the two regions, whereas all orbits in the lower region tend to one of the equilibria.*

Remark 4.2.1. *For the case (3), the boundary separating two regions is the stable manifold connecting neighboring saddle points, i.e., the running periodic orbit. Proposition 4.1 implies that the ω-limit sets consist of equilibria or running periodic orbits.*

4.2.2 Critical regime ($Kr_0 = \Omega$)

In this regime, x also satisfies the same trigonometric equation as that of the Kuramoto model without inertia. Thus it follows from [13] that we can get the explicit formula:

$$t = \frac{2}{\Omega \tan \frac{x_0 - \phi_0}{2} - Kr_0} - \frac{2}{\Omega \tan \frac{x(t)-\phi_0}{2} - Kr_0}.$$

Again, the above explicit formula is not much useful for finding the invariant measure for the fast system. For the critical coupling case, two equilibria x_1^e and x_2^e in subcritical case coalesce and become

$$(x_{3n}^e, y_{3n}^e) = (\phi_0 + 2\tan^{-1}\left(\frac{Kr_0}{\Omega}\right) + n\pi, 0), \quad n = 0, \pm 1, \cdots .$$

The large-time behavior of the general solution can be summarized as follows.

Proposition 4.2.2. [48] *Suppose that the Ω and K satisfy*

$$\frac{\Omega}{Kr_0} = 1.$$

Then there exists a positive number $\sigma_* = \sigma_*\left(\frac{\Omega}{Kr_0}\right) > 0$ such that

1. *For $0 < \frac{1}{Kr_0} < \sigma_*$, there exists an exponentially stable running periodic orbit and orbits that approach the running periodic orbit or equilibria.*

2. *For $\frac{1}{Kr_0} > \sigma_*$, every orbit tends to one of the equilibria.*

3. *For $\frac{1}{Kr_0} = \sigma_*$, the state space is split into two regions, i.e., all orbits in the upper region tend to the boundary between the two regions, whereas all orbits in the lower region tend to one of the equilibria. The phase boundary consists of an orbit connecting two neighboring equilibria (continued 2π-periodically).*

4.2.3 Supercritical regime ($Kr_0 < \Omega$)

First, note that in supercritical case we have no equilibrium points. If the orbits of (4.2.1) were bounded, we could apply the standard Poincare- Bendixson Theorem, but unfortunately this is not the case. Hence we recall some results of Levi et al [48] on the existence of running periodic orbits. The existence of a running periodic orbit and its global stability can be summarized in the following proposition. For this, we introduce two positive numbers $L^{\pm}$ as follows.

$$L^{+} := 2(\Omega + Kr_0), \quad L^{-} := \frac{\Omega - Kr_0}{2}.$$

Proposition 4.2.3. [48] *Suppose that Ω and K satisfy the relation:*

$$\Omega > Kr_0.$$

Then there exists a unique running periodic orbit $\mathcal{P}$ to the system (4.2.1)-(4.2.2) *in the strip $\mathbb{R} \times [L^{-}, L^{+}]$, and it is globally asymptotically stable in the sense that all orbits will attract to $\mathcal{P}$ as $t \to \infty$.*

Remark 4.2.2. *As an immediate corollary of Proposition 4.2.3, for any point $z_0 = (x_0, y_0) \in \mathbb{R}^2$,*

$$\omega(z_0) = \mathcal{P}.$$

and the invariant measure for the fast system (4.2.1) *is supported on the inverse image of a periodic orbit:*

$$\nu(d\lambda) = \frac{1}{T} dx^{-1}(d\lambda),$$

where T is the period of the periodic orbit and x^{-1} denotes the inverse of x. For the zero inertia case $m = 0$, the period T is explicitly computable.

4.3 Limit dynamics of order parameters

In this section, we study the evolution of the order parameters as an application of AKST's theory to the fast-slow system (4.1.5). Recall that the limit measure ν_0 in Theorem 2.2.1 and 2.2.2 an invariant measure of the fast system which was characterized in previous section.

First, we define

$$\begin{aligned} U &:= (\theta, \omega, r, \phi) \in \mathbb{R}^{2M+2}, \quad F(U) := (\omega, f(\omega), 0, 0), \\ G(U) &:= \Big(0, \cdots, 0, 0, \cdots, 0, -\sum_{j=1}^{M} \sin(\theta_j - \phi)\omega_j, \frac{1}{r}\sum_{j=1}^{M} \cos(\theta_j - \phi)\omega_j\Big), \end{aligned}$$

where $f(\omega) = (f_1(\omega), \cdots, f_M(\omega))$ is defined by

$$f_i(\omega) := \frac{1}{m}\Big(-\omega_i + \Omega_i - Kr\sin(\theta_i - \phi)\Big).$$

The system (4.1.5) can be written in a compact form:

$$\frac{dU^\varepsilon}{d\tau} = \frac{F(U^\varepsilon)}{\varepsilon} + G(U^\varepsilon). \tag{4.3.1}$$

Recall that the assumption of $(\mathcal{F}2)$ in Section 2.2 requires that the unique solution of

$$\frac{dU^{(0)}}{dt} = F(U^{(0)}), \quad U^{(0)}(0) = U_{in}, \quad t := \frac{\tau}{\varepsilon}, \tag{4.3.2}$$

lies in a compact subset of $\mathbb{R}^{2M+2}$. In fact, it follows that

$$U^{(0)} \in K.$$

The theory assumes $(\mathcal{F}1) - (\mathcal{F}3)$ and the overall system (4.3.1) has unique solutions on some finite interval, say $0 \le \tau \le 1$. This is certainly true for the initial data in $\mathbf{H}(\delta)$:

$$\mathbf{H}(\delta) := \{(\theta, \omega, r, \phi) \ : \ \theta \in \mathbb{R}^M, \quad \omega \in \mathbb{R}^M, \quad 0 < \delta \le r, \quad \phi \in \mathbb{R}\}.$$

The next step is to identify an *orthogonal* measurement $V(U)$ for which

$$\nabla V(U) \cdot F(U) = 0,$$

that is equivalent to

$$\sum_{i=1}^{M} \Big[\frac{\partial V(U)}{\partial \theta_i}\omega_i + \frac{1}{m}\frac{\partial V(U)}{\partial \omega_i}\Big(-\omega_i + \Omega_i - Kr\sin(\theta_i - \phi)\Big)\Big] = 0.$$

It is easy to see that $V(U) = \tilde{V}(r, \phi)$ will suffice as an orthogonal measurement. In particular, we can select V from the projection map of the $(2M+1)$-th component or $(2M+2)$-th component as our two measurements which of course yield orthogonal observables, i.e.,

$$V(U) = r, \phi.$$

By Theorem 2.2.1, the solution of (4.3.1) with initial data $(\theta, \omega, r, \phi) \in \mathbf{H}(\delta)$, defined on $0 \leq \tau \leq 1$, will have a convergent subsequence $U^{\varepsilon_j}(\cdot)$ that converges to a Young measure $\nu_0(\cdot)$ on $[0,1]$ in the sense of Young measures. The value of the limit Young measure is an invariant measure of the fast system (4.3.2). The oscillators of the fast system (4.3.2) are decoupled, so the invariant measure $\nu_0(\tau)$ is a product measure:

$$\begin{aligned} \nu_0(\tau)(d\lambda) = \nu_1(\tau)(d\lambda_1, d\lambda_{M+1}) \otimes \cdots \nu_M(\tau)(d\lambda_M, d\lambda_{2M}) \\ \otimes \nu_{2M+1}(\tau)(d\lambda_{2M+1}) \otimes \nu_{2M+2}(\tau)(d\lambda_{M+2}), \end{aligned} \tag{4.3.3}$$

where $\lambda = (\lambda_1, \cdots, \lambda_{2M+2})$ and each $\nu_i(\tau)(d\lambda_i, d\lambda_{M+i}),\ 1 \leq i \leq M$ is itself an invariant probability measure for the i-th oscillator equation of (4.1.5). Since $\frac{dr}{d\tau} = 0$ and $\frac{d\phi}{d\tau} = 0$ in the fast system,

$$\nu_{2M+1}(\tau) = \delta(\lambda_{2M+1} - r(\tau)), \qquad \nu_{2M+2}(\tau) = \delta(\lambda_{2M+2} - \phi(\tau)).$$

Furthermore by Theorem 2.2.2, we can use the orthogonal observables associated with any measurement V that satisfy the integral equation:

$$\hat{V}(\nu_0(\tau)) = V(U_{in}) + \int_0^\tau \int_{\mathbb{R}^{2M+2}} \nabla V(\lambda) \cdot G(\lambda) \nu_0(s)(d\lambda) ds. \tag{4.3.4}$$

For the choices $V(U) = r,\ V(U) = \phi$, (4.3.4) yields the evolution of r and ϕ:

$$\begin{aligned} r(\tau) &= r(0) \\ &- \sum_{j=1}^M \int_0^\tau \int_{\mathbb{R}^{2M}} \sin(\lambda_j - \phi(s))\lambda_{M+j}\ \nu_1(s)(d\lambda_1, d\lambda_{M+1}) \otimes \cdots \otimes \nu_M(s)(d\lambda_M, d\lambda_{2M}) ds, \\ \phi(\tau) &= \phi(0) \\ &+ \sum_{j=1}^M \int_0^\tau \int_{\mathbb{R}^{2M}} \frac{1}{r(s)} \cos(\lambda_j - \phi(s))\lambda_{M+j}\ \nu_1(s)(d\lambda_1, d\lambda_{M+1}) \otimes \cdots \otimes \nu_M(s)(d\lambda_M, d\lambda_{2M}) ds. \end{aligned}$$

Next, we substitute (4.3.3) into the equations above to obtain

$$\begin{aligned} r(\tau) = r(0) - \sum_{j=1}^M \int_0^\tau \int_{\mathbb{R}^2} \sin(\lambda_j - \phi(s))\lambda_{M+j}\ \nu_j(s)(d\lambda_j, d\lambda_{M+j}) ds, \\ \phi(\tau) = \phi(0) + \sum_{j=1}^M \int_0^\tau \int_{\mathbb{R}^2} \frac{1}{r(s)} \cos(\lambda_j - \phi(s))\lambda_{M+j}\ \nu_j(s)(d\lambda_j, d\lambda_{M+j}) ds. \end{aligned} \tag{4.3.5}$$

The system (4.3.5) produces what is usually known as an amplitude equation for r, although it is actually the coupled system (4.3.5) that determines r.

• Supercritical regime ($|\Omega_i| > Kr$): In this case, there are no equilibrium solutions to the system (3.1.1). Note that the solution $\dot{\theta}(t)$ has only one sign after some time, i.e., there exists T^* such that either $\dot{\theta}(t) \geq 0$ or $\dot{\theta}(t) \leq 0$ for $t \geq T^*$. We conclude with the slow evolution for r and ϕ in the case of $|\Omega_i| > Kr$. In this case, the invariant measure is supported on the inverse image of the periodic orbit:

$$\nu_i(d\lambda_i, d\lambda_{M+i}) = \frac{1}{T_i} d\theta_i^{-1}(\lambda_i), \qquad \text{where } \theta_i^{-1} \text{ denotes the inverse of } \theta_i.$$

Unlike to the case with zero inertia, however we have no explicit representation for the period T_i. However, the explicit representation of T_i is not needed for the following estimate.

$$\begin{aligned}
\mathcal{I} := & \int_{\mathbb{R}^2} \sin(\lambda_i - \phi)\lambda_{M+i}\nu_i(s)(d\lambda_i, d\lambda_{M+i}) \\
&= \frac{1}{T_i}\int_{\mathbb{R}} \sin(\lambda_i - \phi)\dot{\lambda}_i \; d\theta_i^{-1}(\lambda_i), \quad \text{by } \dot{\lambda}_i = \lambda_{M+i} \\
&= \frac{1}{T_i}\int_0^{T_i} \sin(\theta_i(s_i) - \phi)\dot{\theta}_i(s_i) ds_i, \quad \text{by } s_i := \theta_i^{-1}(\lambda_i) \\
&= \frac{1}{T_i}\int_0^{T_i} \Big(\frac{d}{ds_i}\cos(\theta_i(s_i) - \phi)\Big) ds_i \\
&= 0, \quad \text{by the periodicity of } \theta_i.
\end{aligned} \tag{4.3.6}$$

Similarly, we have

$$\mathcal{J} := \int_{\mathbb{R}^2} \cos(\lambda_i - \phi)\lambda_{M+i}\nu_i(s)(d\lambda_i, d\lambda_{M+i}) = 0.$$

• Subcritical and critical regimes ($|\Omega_i| \leq Kr$): It follows from the results in Section 4 that the support of an invariant measure for the fast system is the union of the equilibria and the running periodic orbit. The contribution of $\mathcal{I}$ and $\mathcal{J}$ with invariant measure, the support of which is the running periodic orbit will be zero based on the same calculation as the supercritical case. However, we note that the invariant measure situated on the stable equilibria has the form

$$\nu_i(s)(d\lambda_i, d\lambda_{M+i}) = \delta(\lambda_i - \theta^e_{in}(s)) \otimes \delta(\lambda_{M+i}).$$

Thus if we insert the above ansatz into $\mathcal{I}$ and $\mathcal{J}$ and use the fact

$$\int_{\mathbb{R}} \lambda_{M+i}\delta(\lambda_{M+i})(d\lambda_{M+i}) = 0,$$

then we have

$$\mathcal{I} = 0, \qquad \mathcal{J} = 0.$$

Therefore, for any cases we have

$$r(s) = r(0), \quad \phi(s) = \phi(0), \quad 0 \le s \le 1.$$

Theorem 4.3.1. *The limiting dynamics for the Kuramoto system* (4.1.5) *as* $\varepsilon \to 0$ *is given as follows:*

$$r(\tau) = const, \qquad \phi(\tau) = const, \qquad\quad 0 \le \tau \le 1.$$

Remark 4.3.1. *1. The results of Theorem 4.3.1 can be interpreted as follows. Since* $t = \frac{\tau}{\varepsilon}$*, if* τ *is bounded, then as* $\varepsilon \to 0+$*,* $t \to \infty$*. Theorem 4.3.1 shows that if we look at our original unscaled system for the* t*-interval* $[0, \frac{1}{\varepsilon}]$*, and map the graph of* $r(t), \varphi(t)$ *onto the fixed scaled* τ*-interval* $[0, 1]$*, the graphs of* r *and* ϕ *will be constant. Specifically, the order parameters* r *and* ϕ *are indeed constant as a function of* τ *in the limit as* $\varepsilon \to 0$*.*

2. The complete synchronization problem of Kuramoto oscillators with inertia was studied in [14, 22], while the slow-fast dynamical systems theory for the flocking and synchronization models without inertia were investigated in [35, 43].

4.4 Numerical Simulations

In this section, we provide numerical simulation results which confirm the estimates of the fast flow in previous sections. As mentioned earlier, the order parameters r and ϕ are constant during the fast flow, and we can assume $\phi = 0$ using the phase-transition invariant property. Thus we consider the following second order ODEs:

$$m\ddot{\theta}_i + \dot{\theta}_i = \Omega_i - Kr_0 \sin\theta_i, \quad i = 1, \dots, N. \tag{4.4.7}$$

Since the equations (4.4.7) are decoupled, we set

$$\theta := \theta_i, \quad \omega := \dot{\theta}_i, \quad \Omega := \Omega_i,$$

for notational simplicity. In all simulations, we also set

$$m = 1, \quad \text{and} \quad N = 10,$$

and we use the fourth order Runge-Kutta method.

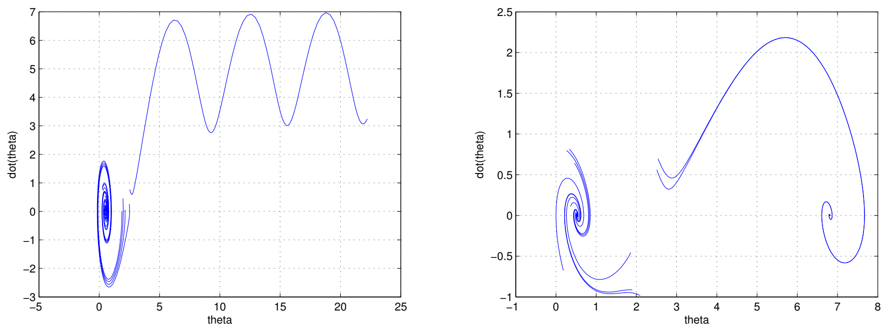
4.4.1 Subcritical case

In this part, we discuss numerical simulations of the fast flow (4.4.7) in the case of $Kr_0 > \Omega$. First, we fix the order parameter r_0 and the natural frequency Ω as follows.

$$r_0 = 0.2, \qquad \Omega = \frac{Kr_0}{2}.$$

Then we change the value of $\frac{1}{Kr_0}$ by varying the coupling strength K.

In Figure 4.1(a), we select the coupling strength $K = 50$ to get a small value $\frac{1}{Kr_0}$, and while in initial configurations, the initial phase θ_0 and frequency ω_0 are chosen randomly from the uniform distributions on $[0, \pi]$ and $[0.1]$, respectively. In this case, we can see that there is a running periodic orbit. In Figure 1(b), the orbits with initial data near the stable equilibrium point tend to that equilibrium point. In contrast, if the initial data is near the unstable equilibrium point, then the orbits with that initial data tend to the stable equilibrium point. In this case, we take the coupling strength $K = 1$ and use the same initial phase θ_0 in Figure 4.1(a), while the initial frequency ω_0 is chosen randomly from the uniform distribution on $[-1, 1]$. This confirms the analytic results (1) and (2) in proposition 4.2.1.



(a) (b)

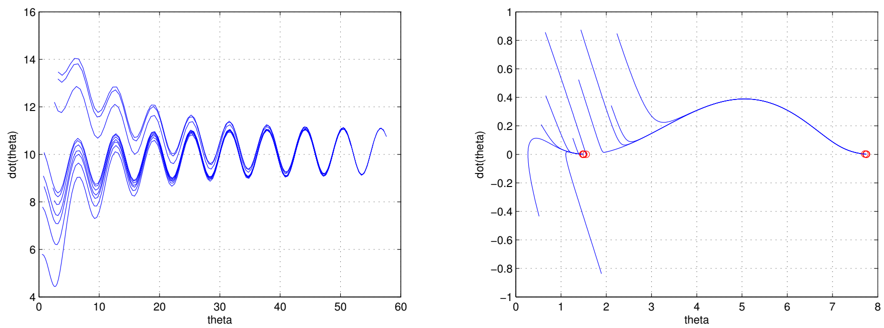
Figure 4.1: Subcritical case ($Kr_0 > \Omega$) : (a) Existence of a running periodic orbit when $0 < \frac{1}{Kr_0} < \sigma_*$. (b) Convergence of each orbit to one of the equilibria when $\frac{1}{Kr_0} > \sigma_*$.

4.4.2 Critical case

In this part, we present some numerical simulations of the fast flow (4.4.7) when $Kr_0 = \Omega$. As the order parameter r_0, we set $r_0 = 0.2$. In Figure 4.2.(a), we select the coupling strength $K = 50$. In the initial configurations, we take the same initial phase θ_0 in Figure 2, and the initial frequency ω_0 is selected from the uniform distribution on $[5, 15]$. In Figure 4.2(b), we choose $K = 1$, while the initial phase θ_0 and frequency ω_0 are selected from the uniform distributions on $[0, \pi]$ and $[-1, 1]$, respectively. The red spots denote the end points of each orbit, and these points coincide with the stable equilibrium points. In Figure 4.2(a),(b), we can see the existence of a running periodic solution and the convergence of each orbit to the one of the equilibria. These numerical simulations are consistent with proposition 4.2.2.

4.4.3 Supercritical case

In this part, we present some numerical simulations of the fast flow (4.4.7) when $Kr_0 < \Omega$. Here the coupling strength $K = 1$ and the initial phase θ_0 and frequency ω_0 are selected from the uniform distributions on $[-\pi, \pi]$ and $[-L,\ L]$, respectively, where $L = |\Omega| + Kr = |\Omega| + 0.2K$. In the supercritical



(a) (b)

Figure 4.2: Subcritical case ($Kr_0 = \Omega$) : (a) Existence of a running periodic orbit when $0 < \frac{1}{Kr_0} < \sigma_*$. (b) Convergence of each orbit to one of the equilibria when $\frac{1}{Kr_0} > \sigma_*$.

case, we select the natural frequency $\Omega = 2Kr_0 = 0.4$ that satisfies $\Omega > Kr_0$. In this setting, we can see that all orbits tend to the unique periodic orbit over time, as shown in Figure 4.3. This result is consistent with Proposition 4.2.3.

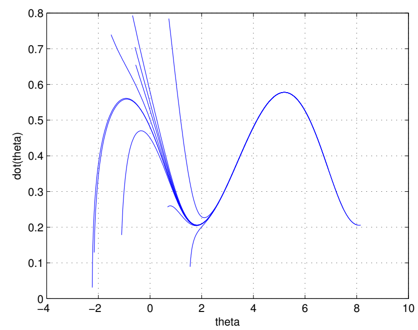
Figure 4.4 confirms that the integral value in (4.3.6) is zero. We selected the initial configurations as follows.

$$r_0 = 0.2, \quad K = 1, \quad \Omega = 1.2058 > Kr_0$$

For the initial phase and frequency, we take

$$\theta_0 = 1.9775, \quad \text{and} \quad \dot{\theta}_0 = -1.6048.$$

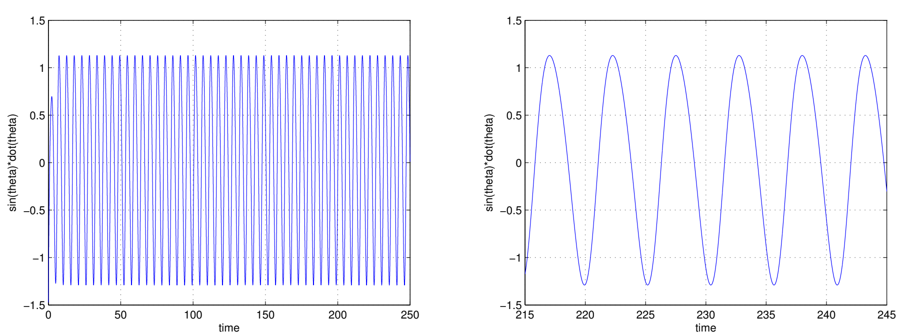
Figure 4.4 (a) shows the time-behavior of $\sin(\theta(t))\dot{\theta}(t)$. In particular, we checked the behavior during the interval $[215, 245]$ in Figure 4.4 (b). In this setting, we see that the maximum value of the function $\sin(\theta(t))\dot{\theta}(t)$ is 0.1848 at $t = 215.9413$, while the time increment that maximizes the value is the same. We summarize the time and integral values in the following table.



(a)

Figure 4.3: Global stability of the running periodic orbit.

maximum value	time	time increment	integral value
0.1848	215.9413		
0.1848	221.1817	5.2404	9.94×10^{-4}
0.1848	226.4221	5.2404	-1.09×10^{-5}
0.1848	231.6625	5.2404	-1.29×10^{-5}
0.1848	236.9029	5.2404	-1.48×10^{-5}
0.1848	242.1433	5.2404	-1.68×10^{-5}



(a) (b)

Figure 4.4: Time-behavior of the function $\sin(\theta(t))\dot{\theta}(t)$

Chapter 5

Planar particle models for flocking and swarming

5.1 A Cucker-Smale type model

In this section, we revisit the C-S type flocking model in [41] and discuss how the model can recast in the framework of AKST's theory.

In the sequel, we set $|| \cdot ||$ to be the standard ℓ_2-norm in $\mathbb{R}^d$:

$$||a|| := \Big(\sum_{i=1}^{d} |a_i|^2 \Big)^{\frac{1}{2}}, \quad a = (a_1, \cdots, a_d) \in \mathbb{R}^d.$$

Let $(x_i(t), v_i(t)) \in \mathbb{R}^{2d}$ be the phase-space coordinate of the i-th particle at time t, and its dynamics is governed by the following singularly perturbed ODE system: For $i = 1, \cdots, N$,

$$\begin{aligned} \frac{dx_i}{dt} &= v_i, \quad t > 0, && (5.1.1) \\ \frac{dv_i}{dt} &= \frac{K}{N\varepsilon} \sum_{j=1}^{N} \psi(||x_j - x_i||)(v_j - v_i) + g_i(v), \end{aligned}$$

subject to initial data:

$$(x_i, v_i)(0) = (x_{i0}, v_{i0}). \tag{5.1.2}$$

Here K is a positive constant representing the strength of coupling and g_i may incorporate friction forces, and ψ is a communication weight between particles satisfying

$$\psi(s) > 0, \quad s \geq 0. \tag{5.1.3}$$

For the simplicity of presentation, we set

$$x := (x_1, \cdots, x_N)^t, \quad v := (v_1, \cdots, v_N)^t \in \mathbb{R}^{Nd}.$$

In order to use the notation of Section 2.2, we take

$$U = \begin{pmatrix} x \\ v \end{pmatrix}, \qquad F(U) = \begin{pmatrix} 0 \\ f(x,v) \end{pmatrix}, \quad G(U) = \begin{pmatrix} v \\ g(v) \end{pmatrix},$$

where $f = (f_1, \cdots, f_N)^T$, $g = (g_1, \cdots, g_N)^T$:

$$f_i(x,v) := \frac{K}{N} \sum_{j=1}^{N} \psi_{ji}(v_j - v_i), \qquad \psi_{ji} := \psi(||x_j - x_i||).$$

U_ε denotes the solution of the system (5.1.1),(5.1.2) and the system can be written in the following convenient form:

$$\frac{dU_\varepsilon}{dt} = \frac{F(U_\varepsilon)}{\varepsilon} + G(U_\varepsilon), \quad t > 0, \qquad U_\varepsilon(0) = U(0). \tag{5.1.4}$$

Note that the fast and slow part of the system (5.1.1),(5.1.2) are as follows.

$$\text{(a)}: \quad \frac{dU}{dt} = F(U) \ : \ \text{fast part}, \qquad \text{(b)}: \quad \frac{dU}{dt} = G(U) \ : \ \text{slow part}. \tag{5.1.5}$$

In order to apply AKST's theory, we need to prepare three ingredients: identification of orthogonal observables, uniform boundedness of fast flow and characterization of ω-limit set of the fast flow.

Lemma 5.1.1. *For each $i = 1, \cdots, N$, $V(x,v) = x_j$ is an orthogonal observable.*

Proof. Recall that to construct an orthogonal observable, we need a measurement V which remains constant under the fast-flow (5.1.5)a. Hence we compute

$$\begin{aligned}\frac{d}{dt}V(U) &= \nabla_U V(U)\cdot\frac{dU}{dt}\\ &= \nabla_U V(U)\cdot F(U)\\ &= \Big(\frac{\partial V}{\partial x_1},\cdots,\frac{\partial V}{\partial x_N},\frac{\partial V}{\partial v_1},\cdots,\frac{\partial V}{\partial v_N}\Big)\cdot\Big(0,\cdots,0,f_1,\cdots,f_N\Big)\\ &= f_1\frac{\partial V}{\partial v_1}+...+f_N\frac{\partial V}{\partial v_N}.\end{aligned}$$

Thus if $V=V(x,v)$ is independent of v_i's, i.e., $V=\hat{V}(x)$, then V becomes the desired measurement, for example,

$$V(U)=x_j,\quad \text{for each } j\in\{1,\cdots,N\}.$$

$\square$

Lemma 5.1.2. (Uniform boundedness of fast flow) *The solution of the fast system* (5.1.5)*a is uniformly bounded, i.e., for any solution* $(x_i,v_i)\in\mathbb{R}^{2d}$ *to the fast-system* (5.1.5)*a, we have*

$$\max_{1\le i\le N}\sup_{t\ge 0}(||x_i(t)||+||v_i(t)||)\le C<\infty,$$

where C is a positive constant only depending on initial data U_0 independent of time t.

Proof. (i) (Uniform boundedness of x): Since $\frac{dx_i}{dt}=0$, we have

$$x_i(t)=x_{i0},\quad \text{for all } t.$$

(ii) (Uniform boundedness of v): We use the standard energy estimate to find

$$\begin{aligned}
\frac{d}{dt}\sum_{i=1}^{N}||v_i||^2 &= \sum_{i=1}^{N} 2v_i\cdot\frac{dv_i}{dt} \\
&= \frac{2K}{N}\sum_{i,j=1}^{N} v_i\cdot\psi_{ji}(v_j-v_i) \\
&= \frac{N}{K}\Big(\sum_{i,j=1}^{N} v_i\cdot\psi_{ji}(v_j-v_i)+\sum_{i,j=1}^{N} v_j\cdot\psi_{ij}(v_i-v_j)\Big) \\
&= -\frac{K}{N}\sum_{i,j=1}^{N}\psi_{ji}||v_j-v_i||^2 \\
&\le 0,
\end{aligned}$$

where we used the symmetry of ψ_{ji}. From this, we obtain

$$||v_i(t)||^2 \le \sum_{i,j=1}^{N}||v_i(t)||^2 \le \sum_{i,j=1}^{N}||v_{i0}||^2 \quad \text{for all } 1\le i\le N \text{ and } t\ge 0.$$

We finally combine the estimates in (i) and (ii) to find the desired result. □

For later use, we set

$$(x,v) := (x_1,\cdots,x_N,v_1,\cdots,v_N)\in\mathbb{R}^{2dN}.$$

Lemma 5.1.3. *For any initial data $U_0 := (x_0,v_0)\in\mathbb{R}^{2dN}$ for the fast system* (5.1.5)*a, we have*

$$\omega(U_0) = \{(x,v)\in\mathbb{R}^{2dN} \ : \ \textit{equilibrium points of } (5.1.5)a\ ,\ \textit{i.e. } x_i = x_{i0}, \quad v_i = v_j, \quad \forall\, i,j\}.$$

Proof. Note that $\mathcal{V} := \sum_{i=1}^{N}||v_i||^2$ is a Lyapunov function and it satisfies

$$\frac{d\mathcal{V}}{dt} = -\frac{K}{N}\sum_{i,j=1}^{N}\psi_{ji}||v_j-v_i||^2. \tag{5.1.6}$$

On the other hand, it follows from LaSalle's invariance principle (see [38], Chapter 10) that

$$\omega(U_0)\subset \text{the largest invariant set in } \Big\{(x,v)\ :\ \frac{d}{dt}\sum_{i=1}^{N}||v_i||^2 = 0\Big\}.$$

Since if

$$\frac{d}{dt}\sum_{i=1}^{N}||v_i||^2 = 0, \quad \text{for all } t,$$

we have

$$x_i = x_{i0}, \qquad \psi(||x_{j0} - x_{i0}||)||v_j - v_i||^2 = 0 \quad \text{for all } t,$$

and hence

$$v_j = v_i \quad \text{and} \quad \frac{dv_j}{dt} = 0.$$

Thus application of LaSalle's theorem proves the lemma. □

Theorem 5.1.1. *The limiting dynamics as $\varepsilon \to 0$ for the C-S system* (5.1.1),(5.1.2) *on $0 \leq t \leq 1$ is that all particles move with same velocity, i.e. they move as a flock:*

$$\frac{d\bar{x}_i(t)}{dt} = \bar{v}_1(t), \quad i = 1, \cdots, N.$$

Proof. For each $j \in \{1, \cdots, N\}$, we take the measurement $V(U) = x_j$:

$$\langle \lambda_j, \mu_0(t)(d\lambda)\rangle - x_{j0} = \int_0^t \langle \lambda_{N+j}, \mu_0(\tau)(d\lambda)\rangle d\tau,$$

which is equivalent to

$$\int_{\mathbb{R}^{2N}} \lambda_j \; \mu_0(t)(d\lambda) - x_{j0} = \int_0^t \int_{\mathbb{R}^{2N}} \lambda_{N+j} \; \mu_0(\tau)(d\lambda)d\tau.$$

In this case, for fixed t, support of μ_0 is contained on the singleton set $\{\overline{x}_1, \cdots, \overline{x}_N, \overline{v}_1, \cdots, \overline{v}_1\}$ by lemma (5.1.3). We can write this again

$$\mu_0(t)(d\lambda) = \delta_t(\lambda_1 - \overline{x}_1)\cdots\delta_t(\lambda_N - \overline{x}_N)\delta_t(\lambda_{N+1} - \overline{v}_1)\cdots\delta_t(\lambda_{2N} - \overline{v}_1).$$

Here δ denotes the Dirac delta measure. Therefore, we obtain the equation

$$\overline{x}_j(t) - \overline{x}_{j0} = \int_0^t \overline{v}_1(\tau) \; d\tau.$$

Since the Young measure is uniquely determined by initial data and is in fact independent of the initial data, we may differentiate the integral expression. □

5.2 A Newtonian model for swarms with Rayleigh friction

In this section, we will consider another particle model which is a Newtonian system with force potential φ and Rayleigh friction. The model has been used by Chuang et al [11], D'Orsogna et al [27], Dunkel et al [28] and related model has been studied by Schweitzer et al [58]. However as far as we know, no one has considered the small mass singular limit of these models. In this section, we pursue this goal.

5.2.1 Description of model system

Consider the particle model on $\mathbb{R}^2$: For $i = 1, \cdots, N$,

$$\begin{aligned} \frac{dx_i}{dt} &= v_i \\ \varepsilon\frac{dv_i}{dt} &= -\sum_{j\neq i}\nabla_x\varphi(x_j - x_i) + \delta(1 - ||v_i||^2)v_i, \end{aligned} \tag{5.2.1}$$

where φ is a pairwise interaction potential and δ is a positive constant relating the strength of self-accelerating force and self-decelerating force, respectively. As noted in the introduction, this model has been the focus of a series of analytical and numerical investigations in the mathematical theory of swarms. We set

$$f_i(x, v) := -\sum_{j\neq i}\nabla_x\varphi(x_j - x_i) + \delta(1 - ||v_i||^2)v_i, \quad g_i(v) := 0,$$

and

$$U = \begin{pmatrix} x \\ v \end{pmatrix}, \qquad F(U) = \begin{pmatrix} 0 \\ f(x, v) \end{pmatrix}, \quad G(U) = \begin{pmatrix} v \\ 0 \end{pmatrix},$$

where $f = (f_1, \cdots, f_N)^T$. Then the system (5.2.1) can be rewritten in compact form:

$$\frac{dU}{dt} = \frac{F(U)}{\varepsilon} + G(U). \tag{5.2.2}$$

Note that the fast flow is given by

$$\begin{aligned} \frac{dx_i}{dt} &= 0, \\ \frac{dv_i}{dt} &= -\sum_{j\neq i} \nabla_x \varphi(x_j - x_i) + \delta(1 - ||v_i||^2)v_i, \end{aligned} \tag{5.2.3}$$

or equivalently

$$\frac{dx_i}{dt} = 0, \quad \frac{dv_i}{dt} = \delta(1 - ||v_i||^2)v_i + c_i, \tag{5.2.4}$$

where c_i satisfies the zero sum relation:

$$c_i := -\sum_{j\neq i} \nabla_x \varphi(x_j - x_i), \qquad \sum_{i=1}^N c_i = 0. \tag{5.2.5}$$

Once c_i is determined by the position $x = (x_1, \cdots, x_N)$, then the dynamics for each v_i is completely decoupled. We next show that the fast flow is uniformly bounded.

Lemma 5.2.1. *Let $v_i = v_i(t)$ be the solution to the system* (5.2.3). *Then we have*

$$||v_i(t)|| \leq \max\{||v_{i0}||, R\}, \quad t \geq 0,$$

where D is a positive constant:

$$R := \Big(\frac{\delta + ||c_i||}{\delta}\Big)^{\frac{1}{2}}.$$

Proof. We consider the two cases depending on initial data:

$$\text{either } v_{i0} \in B_R(0), \qquad \text{or} \qquad v_{i0} \in B_R(0)^c,$$

where $B_R(0)$ is the ball with a center 0 and radius R.

Case 1 ($v_{i0} \in B_R(0)^c$): Since $R > 1$, as long as $v_i(t) \in B_R(0)^c$, $||v_i||^2$ is strictly decreasing along the fast flow (4.2.1). This can be seen as follows. We take the inner product in (5.2.3) with v_i to find

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}||v_i||^2 &= \delta(1 - ||v_i||^2)||v_i||^2 - c_i \cdot v_i \\ &\leq -\delta||v_i||^4 + (\delta + ||c_i||)||v_i||^2 \\ &< 0, \end{aligned}$$

where we used the fact that $||v_i|| \geq 1$. Hence $||v_i||$ is strictly decreasing, i.e.,

$$||v_i(t)|| \leq ||v_{i0}||, \quad t \geq 0.$$

Case 2 ($v_{i0} \in B_R(0)$): In this case, we can use the result of Case 1 as follows. If $||v_i(t)|| \leq R,\ t \geq 0$, then we are done. However if there exists some finite time t_0 such that

$$||v_i(t_0)|| = R,$$

then at that instant $t = t_0$, we have

$$\frac{1}{2}\frac{d}{dt}||v_i||^2\Big|_{t=t_0} \leq 0.$$

Hence the flow cannot exit the ball $B_R(0)$. We finally combine Case 1 and Case 2 to get the desired result. □

Of course, an important consequence of Lemma 5.2.1 is that equations (5.2.3) for v_i (with fixed c_i) are decoupled and hence the Poincare-Bendixson Theorem applies to each individual equation.

Lemma 5.2.2. *(i) If $c_i \neq 0\ \forall\ i$, the fast system* (5.2.3) *has at most 3^N number of equilibrium solutions.*
(ii) If $c_i = 0$, then either $v_{ei} = 0$ or any points on the unit sphere $||v_{ei}||$ are equilibrium points.

Proof. Note that an equilibrium solution $(v_{e1}, \cdots, v_{eN})$ satisfies

$$\delta(1 - ||v_{ei}||^2)v_{ei} - c_i = 0. \tag{5.2.6}$$

Once we know $||v_{ei}|| \neq 1$, then v_{ei} is given by

$$v_{ei} = \frac{c_i}{\delta(1 - ||v_{ei}||^2)}, \quad \text{if } ||v_{ei}|| \neq 1, \tag{5.2.7}$$

i.e., nonunit $||v_{ei}||$ uniquely determines the equilibrium solution v_{ei}. It follows from (5.2.6) that

$$(1 - ||v_{ei}||^2)^2||v_{ei}||^2 = \frac{||c_i||^2}{\delta^2}. \tag{5.2.8}$$

Note that $||v_{ei}||^2$ is given by the intersection points between two functions

$$(1-y)^2 y, \ y \geq 0, \quad \text{and} \quad \frac{||c_i||^2}{\delta^2}.$$

By direct calculation, it is easy to see that the function $f(y) = (1-y)^2 y$ has local extreme at $x = 0, \frac{1}{3}, 1$, and

$$f(0) = 0, \quad f\Big(\frac{1}{3}\Big) = \frac{4}{27}, \quad f(1) = 0.$$

Note that if $||c_i|| \neq 0$, it is easy to see that $||v_{ei}|| \neq 1$. Hence the determination of $||v_{ei}||$ yields the equilibrium solution.

Case 1 ($\frac{||c_i||^2}{\delta^2} > \frac{4}{27}$): By the graphical argument, we have the unique solution v_{11} satisfying

$$||v_{11}|| > 1.$$

Case 2 ($\frac{||c_i||^2}{\delta^2} = \frac{4}{27}$): In this case we have two roots for (5.2.8) satisfying

$$||v_{21}|| = \frac{1}{\sqrt{3}} \quad \text{and} \quad ||v_{22}|| > 1.$$

Case 3 ($0 < \frac{||c_i||^2}{\delta^2} < \frac{4}{27}$): In this case we have three roots for (5.2.8) satisfying

$$||v_{31}|| < \frac{1}{\sqrt{3}}, \quad \frac{1}{\sqrt{3}} < ||v_{32}|| < 1, \quad \text{and} \quad ||v_{33}|| > 1.$$

Case 4 $||c_i|| = 0$: In this case, we have infinitely many solutions, i.e.,

$$||v_{41}|| = 0, \quad ||v_{42}|| = 1.$$

□

We next study the stability of the equilibria obtained in Lemma 5.2.2. For a given equilibrium point v_e for the system (5.2.3), we introduce a perturbation $\hat{v}_i$:

$$\hat{v}_i := v_i - v_e.$$

Then the linearized system for (5.2.3) is

$$\frac{d\hat{v}_i}{dt} = (-2\delta v_e \cdot \hat{v}_i)v_e + (\delta - \delta||v_e||^2)\hat{v}_i,$$

or equivalently,

$$\frac{d}{dt}\begin{pmatrix} \hat{v}_{i1} \\ \hat{v}_{i2} \end{pmatrix} = \begin{pmatrix} \delta - \delta(3v_{e1}^2 + v_{e2}^2) & -2\delta v_{e1}v_{e2} \\ -2\delta v_{e1}v_{e2} & \delta - \delta(v_{e1}^2 + 3v_{e2}^2) \end{pmatrix}\begin{pmatrix} \hat{v}_{i1} \\ \hat{v}_{i2} \end{pmatrix}. \qquad (5.2.9)$$

By direct calculation, it is easy to see that the above linear system has two real eigenvalues:

$$\lambda_1 = \delta - \delta(v_{e1}^2 + v_{e2}^2), \qquad \lambda_2 = \delta - 3\delta(v_{e1}^2 + v_{e2}^2).$$

5.2.2 Classification of equilibria

In this part, we discuss different cases above to list the equilibria and their stability.

• Stability of equilibria:

Case 1 $\left(\frac{||c_i||^2}{\delta^2} > \frac{4}{27}\right)$:

$$v_{11},\ (\lambda_2 < \lambda_1 < 0):\ \text{stable node.}$$

Case 2 $\left(\frac{||c_i||^2}{\delta^2} = \frac{4}{27}\right)$:

v_{21}, $(0 = \lambda_2 < \lambda_1)$: unstable degenerate node; v_{22}, $(\lambda_2 < \lambda_1 < 0)$: stable node.

Case 3 $\left(0 < \frac{||c_i||^2}{\delta^2} < \frac{4}{27}\right)$:

v_{31}, $(0 = \lambda_2 < \lambda_1)$: unstable node; v_{32}, $(\lambda_2 < 0 < \lambda_1)$: unstable saddle;
v_{33} $(\lambda_2 < \lambda_1 < 0)$: stable node.

• Possible configurations of phase portrait:

In each case, the Poincare-Bendixson Theorem and the index of an isolated critical point may be used to describe the possible configuration of the phase portrait as follows.

Case 1: We have a stable node with index 1 which does not preclude a periodic orbit around the node.

Case 2: We may have a stable node v_{22}, with a trajectory joining critical points or a periodic orbit around v_{22} or neither of these two possibilities. The index theory does not apply to the degenerate equilibrium point v_{21}.

Case 3: We may have the saddle point connected to itself by a homoclinic orbit, connected to the unstable node, or connected to the stable node. Furthermore index theory allows for a periodic solution around he unstable node v_{31}, the stable node v_{33}, or around all these equilibria.

5.2.3 Limit dynamics of the system (5.2.3)

Recall the Young measure associated with the limit of the fast flow must be an invariant measure supported on the ω-limit set of the fast motion. This allows us to finally record the possible support of this measure in all these cases.

Case 1. The support of the invariant measure is the union of a critical point and the possible periodic orbit.

Case 2. The support of the invariant measure is the union of the two critical points and the possible periodic orbit.

Case 3. The support of the invariant measure is the union of the three critical points and the two possible periodic orbits.

As in Section 5.1, we can take $x_j,\ j = 1, \cdots, N$ to be measurements and immediately record the limit dynamics. The main result of this section is the following description of the limit dynamics of the system (5.2.1).

Theorem 5.2.1. *The limit dynamics of* (5.2.1) *as* $\varepsilon \to 0$ *for say* $0 \leq t \leq 1$ *is given by*

$$x_j(t) = x_j(0) + \int_0^t \int_{\mathbb{R}^{2N}} \lambda_{N+j}\mu_0(\tau) d\lambda d\tau, \tag{5.2.10}$$

where the support of $\mu_0(\tau)$ may change as

$$c_i = -\sum_{j \neq i} \nabla_x \varphi(x_j - x_i), \quad i = 1, \cdots, N,$$

evolves between Case 1, 2, 3.

While theorem 5.3.1 is quite general and goes beyond the theorems for the $N = 2$ case given by Schweitzer-Ebeling-Tilch [58] and Dunkel-Ebeling-Erdmann [28]. The restriction

$$\sum_{i=1}^{N} c_i = 0$$

has important consequences. In terms of v_{ei}, this means

$$\sum_{i=1}^{N} v_{ei}(1 - ||v_{ei}||^2) = 0.$$

For example this says we cannot have all the first components v_{ei}^1 of v_{ei} with the same sign and remain in Case 1 (and a similar statement for v_{ei}^2). Thus we conclude,

(i) there must be a switch in orientation of the vector fields v_{ei} for some value of i, or
(ii) for some i, we are in Case 2 or Case 3.

If (i) occurs and the initial data v_{i0} is sufficiently close to the unique equilibrium, the Young measure is supported on that equilibrium point. The limit motion is simply

$$\frac{dx_i}{dt} = v_{ie}(x), \quad i = 1, \cdots, N, \tag{5.2.11}$$

where there must be a switch in orientation of at least one vector field $v_{ie}(x)$. In particular, for the case $N = 2$, the two vector fields must reverse orientation. These are seen in the numerical results in Figure 7 of Schweitzer-Ebeling-Tilch [58] and Figure 1 of Chuang et al [11]. Furthermore Schweitzer-Ebeling-Tilch assert that when we move beyond Case 1 "the swarm does not establish an internal order". We amend their remark: Our Theorem 5.2.1 says while the internal order may be complicated, it is completely determined by the three specific cases of the theorem.

5.3 Numerical Simulations

In this section, we provide a numerical example to illustrate Theorem 5.2.1. In an simulations, we choose the force potential φ to be of Morse type as in [27]:

$$\varphi(||x_i - x_j||) = -C_a e^{-\frac{|x_i - x_j|}{l_a}} + C_r e^{-\frac{|x_i - x_j|}{l_r}},$$

where C_a and C_r are the strength of attraction and repulsion respectively, and l_a and l_r are the effective interaction length scales. In fact, the model introduced in [27] is as follows:

$$\begin{aligned} \frac{dx_i}{dt} &= v_i, \quad i = 1, \cdots, N, \\ m\frac{dv_i}{dt} &= \alpha v_i - \beta ||v_i||^2 v_i - \nabla_{x_i}\varphi(||x_i - x_j||). \end{aligned} \tag{5.3.12}$$

To compare (5.3.12) with (5.2.1), we introduce a change of variables:

$$\tilde{x} := \sqrt{\frac{\beta}{\alpha}}x, \quad \tilde{v} := \sqrt{\frac{\beta}{\alpha}}v.$$

Note that the interaction term can be written as follows

$$\nabla_{x_i}\varphi(||x_i - x_j||) = \Big(\frac{\partial \tilde{x}_i}{\partial x_i}\Big)\cdot\nabla_{\tilde{x}_i}\varphi\Big(\sqrt{\frac{\alpha}{\beta}}||\tilde{x}_j - \tilde{x}_i||\Big) = \sqrt{\frac{\beta}{\alpha}}\nabla_{\tilde{x}_i}\varphi\Big(\sqrt{\frac{\alpha}{\beta}}||\tilde{x}_j - \tilde{x}_i||\Big),$$

and we have

$$\begin{aligned} \frac{d\tilde{x}_i}{dt} &= \tilde{v}_i(t), \quad i = 1, \cdots, N, \\ m\frac{d\tilde{v}_i}{dt} &= \alpha(1 - ||\tilde{v}_i||^2)\tilde{v}_i - \frac{\beta}{\alpha}\sum_{j\neq i}\nabla_{\tilde{x}_i}\varphi\Big(\sqrt{\frac{\alpha}{\beta}}||\tilde{x}_i - \tilde{x}_j||\Big), \end{aligned} \tag{5.3.13}$$

which is of the form (5.2.1). For numerical simulations, we employed the 4th-order Runge-Kutta method as a numerical scheme. We explain the numerical simulation performed in Figure. For the simulations, all parameters are taken as follows:

$$C_a = 0.5, \quad C_r = 1, \quad l_a = 2, \quad l_r = 0.5, \quad \alpha = 1.6, \quad \beta = 0.5, \quad N = 100.$$

These parameters are used in [27]. The v_{ei} satisfies the equation.

$$\delta(1 - ||v_{ei}||^2)v_{ei} - c_i = 0,$$

where

$$\begin{aligned} c_i &= \frac{\beta}{\alpha}\sum_{j\neq i}\nabla_{\tilde{x}_i}\varphi(\sqrt{\frac{\alpha}{\beta}}||\tilde{x}_i - \tilde{x}_j||) \\ &= \sqrt{\frac{\beta}{\alpha}}\sum_{j\neq i}\Big(\frac{C_a}{l_a}e^{-\frac{\sqrt{\frac{\alpha}{\beta}}||\tilde{x}_i-\tilde{x}_j||}{la}} - \frac{C_r}{l_r}e^{-\frac{\sqrt{\frac{\alpha}{\beta}}||\tilde{x}_i-\tilde{x}_j||}{lr}}\Big)\cdot\frac{\tilde{x}_i - \tilde{x}_j}{||\tilde{x}_i - \tilde{x}_j||}. \end{aligned}$$

and hence

$$||v_{ei}||^2(1 - ||v_{ei}||^2)^2 = \frac{||c_i(\tilde{x}(t))||^2}{\delta^2},$$

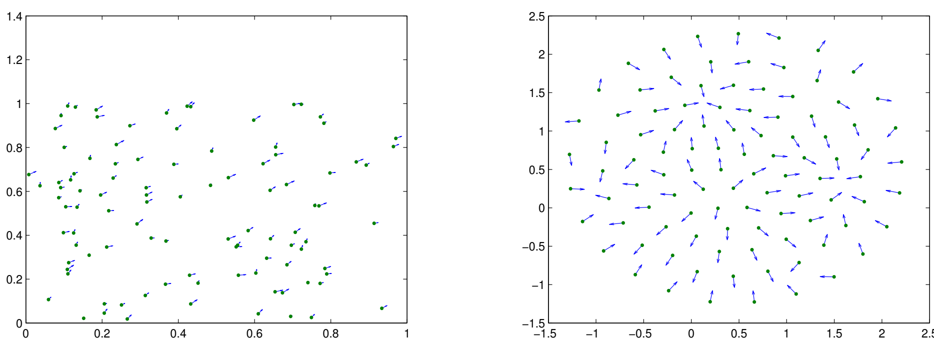
i.e. $||v_{ei}||^2$ is the root of cubic equation $x^3 - 2x^2 + x = \frac{||c_i(\tilde{x}(t))||^2}{\delta^2}$. So by Cardano's formula, $||v_{ei}||^2$ has just one real root:

$$\begin{aligned} ||v_{ei}||^2 &= \frac{2}{3} + \Big\{\frac{c}{2\delta^2} - \frac{1}{27} + \sqrt{\frac{c}{4\delta^4}\Big(c - \frac{4\delta^2}{27}\Big)}\Big\}^{\frac{1}{3}} + \Big\{\frac{c}{2\delta^2} - \frac{1}{27} - \sqrt{\frac{c}{4\delta^4}\Big(c - \frac{4\delta^2}{27}\Big)}\Big\}^{\frac{1}{3}} \\ &= \frac{2}{3} + \Big(\frac{\sqrt{c}}{2\delta} + \frac{1}{2\delta}\sqrt{c - \frac{4\delta^2}{27}}\Big)^{\frac{2}{3}} + \Big(\frac{\sqrt{c}}{2\delta} - \frac{1}{2\delta}\sqrt{c - \frac{4\delta^2}{27}}\Big)^{\frac{2}{3}} \\ &= \Big\{\Big(\frac{\sqrt{c}}{2\delta} + \frac{1}{2\delta}\sqrt{c - \frac{4\delta^2}{27}}\Big)^{\frac{1}{3}} + \Big(\frac{\sqrt{c}}{2\delta} - \frac{1}{2\delta}\sqrt{c - \frac{4\delta^2}{27}}\Big)^{\frac{1}{3}}\Big\}^2, \end{aligned}$$

where $c := ||c_i(\tilde{x}(t))||^2$. Therefore the limit equation (5.3.13) in Case1 is

$$\frac{d\tilde{x}_i}{dt} = \frac{c_i(\tilde{x}(t))}{\delta(1 - ||v_{ei}||^2)},$$

where $\delta = 1.6$. We choose random initial data and display the solution at t=20:



(a) $t = 0$ (b) $t = 20$

Chapter 6

Conclusion and future project

We provided the asymptotic formation of phase-locked states arising from initial configuration θ^0 via the Kuramoto model. Our first estimate(Theorem 3.2.1) deals with the dynamic formation of phase-locked states from initial configurations distributed over the half circle. We show that initial configurations along the Kuramoto flow undergo two dynamic stages (transition and relaxation):

- (Transition stage): Initial phase configurations whose diameters are greater than $\frac{\pi}{2}$ are attracted to the stable region in a finite-time, which is determined by the diameter of initial phase configurations.
- (Relaxation stage): After configurations reach to the stable regions, they asymptotically relax to phase-locked states.

Of course, once initial configurations are already in a stable region, transition stage does not occur.

Our second result(Theorem 3.3.1 and 3.3.2) is on the structure of the phase-locked states emerged from initial phase configurations. Theorem 3.3.1 allows us to calculate the exact number of collisions between oscillators. Oscillators with different natural frequencies can collide before they reach phase-locked states. In our current setting, these collisions between two fixed oscillators are at most once. Once the oscillators with larger natural frequencies overtake oscillators with slower natural frequencies, then after that moment, they will not collide any more.

For the Kuramoto oscillators with inertia, the fast part of the system consists of N-decoupled pendulum equations with constant friction and torque as the phase of individual oscillators, whereas the slow part governs the evolution of order parameters that represent the amplitude and phase of the centroid of the oscillators. We show that Kuramoto's order parameters become stationary regardless of the coupling strength(theorem 4.3.1).

In a Cucker-Smale type model, we obtain the limiting dynamics as $\varepsilon \to 0$. We note that the result of Theorem 5.1.1 improves the corresponding results in [41] where the restrictive assumptions $\psi \geq \psi^* > 0$ and non-increasing property of ψ in its argument are crucial to get the flocking estimate. Of course here, our result is much weaker in that we have only weak$*$-convergence to our limit equations of Theorem 5.1.1. In this part, we use the LaSalle's invariance principle and positivity of the communication weight to obtain the support of the invariant measures.

The last part of this thesis is about the Newtonian type model with Rayleigh friction. Un like the previous C-S type model, this model can be describe the milling phenomena. Since we consider this model in plane, Poincare-Bendixson Theorem and the index of an isolated critical point are useful to classify the support of invariant measures. The limit dynamics of this model can be decided by the cases of the support of invariant measures.

In this thesis, thanks to the AKST's approach, we can try to extend the previous results. However, that limit dynamics is the weak-* sense limit behavior. So if we obtain the limit dynamics in the strong sense, we have to find the another approach instead of the singular perturbation theory.

Chapter 7

Appendix

7.1 Appendix A

7.1.1 The proof of Proposition 3.2.1

In this subsection, we present the proof of Proposition 3.2.1.

For the case $\mathcal{D}(\theta^0) < \frac{\pi}{2}$, we have

$$\mathcal{D}^\infty = \mathcal{D}(\theta^0),$$

and it follows from Lemma 3.2.1 that

$$\mathcal{D}(\theta(t)) \le \mathcal{D}(\theta^0) = \mathcal{D}^\infty, \quad t \ge 0.$$

Next, we consider the following case

$$\mathcal{D}(\theta^0) \in \left(\frac{\pi}{2}, \pi\right).$$

We first show that there exists a finite-time t_0 such that

$$\mathcal{D}(\theta(t_0)) \le \mathcal{D}^\infty \in \left(0, \frac{\pi}{2}\right). \tag{7.1.1}$$

Below we claim:

$$\text{If } \ \mathcal{D}(\theta(t)) \in [\mathcal{D}^\infty, \mathcal{D}(\theta^0)], \qquad \text{then} \quad \frac{d}{dt}\mathcal{D}(\theta(t)) < 0.$$

The proof of claim: Suppose that

$$\mathcal{D}^\infty \le \mathcal{D}(\theta(t)) \le \mathcal{D}(\theta^0) \quad \text{for} \quad t \ge 0. \tag{7.1.2}$$

This yields

$$\sin \mathcal{D}(\theta(t)) \ge \sin \mathcal{D}^\infty = \sin \mathcal{D}(\theta^0), \quad t \ge 0.$$

It follows from (3.2.6) that

$$\begin{aligned}
\dot{\mathcal{D}}_\theta(t) &\le \mathcal{D}(\Omega) - K \sin(\mathcal{D}(\theta(t)) \\
&\le \mathcal{D}(\Omega) - K \sin \mathcal{D}^\infty \\
&= \mathcal{D}(\Omega) - K \sin \mathcal{D}(\theta^0) \\
&= \Big(1 - \frac{K}{K_e}\Big)\mathcal{D}(\Omega) \\
&< 0.
\end{aligned}$$

We integrate the above differential inequality to get

$$\mathcal{D}(\theta(t)) \le \mathcal{D}(\theta^0) + \Big(1 - \frac{K}{K_e}\Big)\mathcal{D}(\Omega)t.$$

Note that

$$\mathcal{D}(\theta^0) + \Big(1 - \frac{K}{K_e}\Big)\mathcal{D}(\Omega)t \le \mathcal{D}^\infty \quad \Longleftrightarrow \quad t \ge \frac{\mathcal{D}(\theta^0) - \mathcal{D}^\infty}{\Big(\frac{K}{K_e} - 1\Big)\mathcal{D}(\Omega)} =: t_0.$$

Hence $\mathcal{D}(\theta(t))$ leaves the interval $[\mathcal{D}^\infty, \mathcal{D}(\theta^0)]$ after the time $t = t_0$.
We next show that

$$\mathcal{D}(\theta(t)) \le \mathcal{D}^\infty < \frac{\pi}{2} \quad \text{for} \quad t \ge t_0.$$

Define

$$\mathcal{T} := \{\ T > t_0 : \mathcal{D}(\theta(t)) < \mathcal{D}^\infty,\ \forall t \in [t_0, T)\} \quad \text{and} \quad T^* := \sup \mathcal{T}.$$

Since $\mathcal{D}(\theta(t))$ is continuous function, there exists T such that

$$\mathcal{D}(\theta(t)) < \mathcal{D}^\infty \quad \text{for all } t \in [t_0, T).$$

Thus $\mathcal{T}$ is nonempty and T^* exists. Below we will show that

$$T^* = \infty.$$

Suppose not, i.e., T^* is finite. Then we will derive the Gronwall's inequality:

$$\dot{\mathcal{D}}(\theta(t)) + \frac{\mathcal{D}(\Omega)}{\mathcal{D}^\infty}\mathcal{D}(\theta(t)) < \mathcal{D}(\Omega), \quad t \in [t_0, T_*). \tag{7.1.3}$$

The derivation of (7.1.3): Since $\dfrac{\sin x}{x}$, $x \in \Big(0, \dfrac{\pi}{2}\Big)$ is strictly decreasing, we have

$$\frac{\sin \mathcal{D}^\infty}{\mathcal{D}^\infty} < \frac{\sin \mathcal{D}(\theta(t))}{\mathcal{D}(\theta(t))} \quad \text{and} \quad t \in \mathcal{T}. \tag{7.1.4}$$

We use (7.1.4) and the differential inequality in Lemma 3.2.3:

$$\dot{\mathcal{D}}(\theta(t)) + K \sin \mathcal{D}(\theta(t)) - \mathcal{D}(\Omega) \le 0, \quad K > K_e = \frac{\mathcal{D}(\Omega)}{\sin \mathcal{D}(\theta^0)},$$

to derive (7.1.3):

$$\begin{aligned}
&\dot{\mathcal{D}}(\theta(t)) + \frac{\mathcal{D}(\Omega)}{\mathcal{D}^\infty}\mathcal{D}(\theta(t)) \\
&\quad < \dot{\mathcal{D}}(\theta(t)) + K\frac{\sin \mathcal{D}^\infty}{\mathcal{D}^\infty}\mathcal{D}(\theta(t)) \\
&\quad \le \dot{\mathcal{D}}(\theta(t)) + K \sin \mathcal{D}(\theta(t)) \\
&\quad \le \mathcal{D}(\Omega), \quad t \in \mathcal{T}.
\end{aligned}$$

By Gronwall's lemma, we have

$$\begin{aligned}
\mathcal{D}(\theta(t)) &< \mathcal{D}(\theta(t_0))e^{-\frac{\mathcal{D}(\Omega)}{\mathcal{D}^\infty}(t-t_0)} + \mathcal{D}^\infty(1 - e^{-\frac{\mathcal{D}(\Omega)}{\mathcal{D}^\infty}(t-t_0)}), \\
&= \mathcal{D}^\infty + (\mathcal{D}(\theta(t_0)) - \mathcal{D}^\infty)e^{-\frac{\mathcal{D}(\Omega)}{\mathcal{D}^\infty}(t-t_0)} \\
&=: f(t), \quad t \in \mathcal{T}.
\end{aligned}$$

On the other hand, since

$$f'(t) = \mathcal{D}(\Omega)\Big(1 - \frac{\mathcal{D}(\theta(t_0))}{\mathcal{D}^\infty}\Big)e^{-\frac{\mathcal{D}(\Omega)}{\mathcal{D}^\infty}(t-t_0)} > 0,$$

$f(t)$ is strictly increasing and the limit of $f(t)$ is clearly $\mathcal{D}^\infty$. Hence we have

$$\mathcal{D}^\infty = \lim_{t \to T^*-} \mathcal{D}(\theta(t)) \le \lim_{t \to T^*-} f(t) = f(T^*) < \lim_{t \to \infty} f(t) = \mathcal{D}^\infty.$$

This is a contradiction and thus we obtain $T^* = \infty$. Hence we have

$$\mathcal{D}(\theta(t)) < \mathcal{D}^\infty, \quad \forall\, t \geq t_0.$$

Finally, for the case where $\mathcal{D}(\theta^0) = \frac{\pi}{2}$, we use the differential inequality (3.2.6) to see

$$\dot{\mathcal{D}}(\theta(0)) \leq \mathcal{D}(\Omega) - K \sin \mathcal{D}(\theta^0) = \mathcal{D}(\Omega) - K < 0,$$

where we used $K > K_e = \frac{\mathcal{D}(\Omega)}{\sin \mathcal{D}(\theta^0)} = \mathcal{D}(\Omega)$. Hence we have

$$\mathcal{D}(\theta(\delta)) < \frac{\pi}{2}, \quad \text{for some } 0 < \delta \ll t_0,$$

and we can choose η satisfies the following

$$K > \frac{\mathcal{D}(\Omega)}{\sin \eta} > \mathcal{D}(\Omega) \quad \text{and} \quad \mathcal{D}(\theta(\delta)) < \eta < \mathcal{D}(\theta^0).$$

We finally use the same argument as the case $\mathcal{D}(\theta^0) \in \Big(0, \frac{\pi}{2}\Big)$ to find

$$\mathcal{D}(\theta(t)) \leq \eta \leq \mathcal{D}^\infty \quad \text{for} \quad t \geq t_0.$$

7.1.2 The proof of Proposition 3.2.2

In this subsection, we provide the proof of proposition 3.2.2.

If $\mathcal{D}^\infty \leq \varepsilon$, then we can take $K > K_e$ by Proposition 3.2.1. So we consider the only following case,

$$\mathcal{D}^\infty > \varepsilon.$$

First, by Proposition 3.2.1, we have

$$\mathcal{D}(\theta(t)) \leq \mathcal{D}^\infty \quad \text{for} \quad t \geq t_0.$$

Then by simple calculation, we obtain

$$\begin{aligned}
\dot{\mathcal{D}}(\theta(t)) &= \Omega_M - \Omega_m + \frac{K}{N} \sum_{j=1}^{N} \big(\sin(\theta_j - \theta_M) - \sin(\theta_j - \theta_m)\big) \\
&\leq \mathcal{D}(\Omega) + \frac{\bar{K}}{N} \sum_{j=1}^{N} \big((\theta_j - \theta_M) - (\theta_j - \theta_m)\big) \\
&\leq \mathcal{D}(\Omega) - \bar{K}\mathcal{D}(\theta(t)) \quad \text{for} \quad t \geq t_0,
\end{aligned}$$

where

$$\bar{K} = \frac{K \sin \mathcal{D}^\infty}{\mathcal{D}^\infty}.$$

Thus we have the following inequality

$$\begin{aligned} \mathcal{D}(\theta(t)) &\leq \Big(\mathcal{D}(\theta(t_0)) - \frac{\mathcal{D}(\Omega)}{\bar{K}}\Big) e^{-\bar{K}(t-t_0)} + \frac{\mathcal{D}(\Omega)}{\bar{K}} \\ &\leq \frac{\varepsilon}{2} + \frac{\mathcal{D}(\Omega)}{\bar{K}} \quad \text{for} \quad t \geq T_*, \end{aligned}$$

where we used the fact that

$$\lim_{t\to\infty} e^{-\bar{K}t} = 0.$$

Note that

$$\frac{\mathcal{D}(\Omega)}{\bar{K}} < \frac{\varepsilon}{2} \quad \Longleftrightarrow \quad K > 2\frac{\mathcal{D}(\Omega)}{\varepsilon}\frac{\mathcal{D}^\infty}{\sin \mathcal{D}^\infty}.$$

Therefore if we choose K sufficiently large such that

$$K > \max\Big\{K_e, \frac{2\mathcal{D}^\infty \mathcal{D}(\Omega)}{\varepsilon \sin \mathcal{D}^\infty}\Big\},$$

we obtain

$$\mathcal{D}(\theta(t)) \leq \varepsilon \quad \text{for} \quad t \geq T_*.$$

7.2 Appendix B

7.2.1 The proof of Lemma 3.3.1

In this subsection, we provide the proof of Lemma 5.1.

(i) It follows from Lemma 3.2.1 that

$$0 < \mathcal{D}(\theta(t)) \le \mathcal{D}_0, \quad t \ge 0.$$

First, by assumption (2), we obtain the following inequality

$$\theta_{ij}(t) > 0, \quad t \in [t_0, T).$$

On the other hand, we use (3.4) and Lemma 3.2.2.(ii) to get the equation for θ_{ij}:

$$\dot{\theta}_{ij} + K\beta_{ij}(N,\theta)\sin\theta_{ij} = \Omega_{ij}, \tag{7.2.1}$$

where the mean-field coupling term β_{ij} is given by

$$\beta_{ij}(N,\theta) := 1 - \frac{1}{N}\sum_{l\neq i,j} C_{ij}^l.$$

We next try to obtain an upper bound for β_{ij}. We claim:

$$\max_{i,j}\beta_{ij}(N,\theta) \le 1 - \frac{N-2}{N}\Big(1 - \frac{1}{\cos\frac{\mathcal{D}(\theta^0)}{2}}\Big) = U(N,\mathcal{D}(\theta^0)). \tag{7.2.2}$$

The proof of claim: Since

$$|\theta_i - \theta_j| \le \mathcal{D}(\theta^0) \in (0,\pi),$$

we have

$$\cos\mathcal{D}(\theta^0) \le \cos\Big(\frac{\theta_l-\theta_i}{2}+\frac{\theta_l-\theta_j}{2}\Big) \le 1, \quad \text{and} \quad 0 < \cos\frac{\mathcal{D}(\theta^0)}{2} \le \cos(\frac{\theta_i-\theta_j}{2}) \le 1.$$

These yield

$$\frac{\cos(\frac{\theta_l-\theta_i}{2}+\frac{\theta_l-\theta_j}{2})}{\cos\frac{\theta_i-\theta_j}{2}} \le \frac{1}{\cos\frac{\mathcal{D}(\theta^0)}{2}}.$$

Thus we have

$$C_{ij}^l \geq 1 - \frac{1}{\cos\frac{\mathcal{D}(\theta^0)}{2}}.$$

Hence we get

$$\beta_{ij}(N, \mathcal{D}(\theta^0)) = 1 - \frac{1}{N}\sum_{l\neq i,j} C_{ij}^l \leq 1 - \frac{N-2}{N}\Big(1 - \frac{1}{\cos\frac{\mathcal{D}(\theta^0)}{2}}\Big) = U(N, \mathcal{D}(\theta^0)).$$

Since i and j are arbitrary, this completes the proof. Finally we combine (7.2.1), (7.2.2) and $\theta_{ij} \geq \sin\theta_{ij}$ to get the Gronwall's inequality for θ_{ij}:

$$\dot{\theta}_{ij} + KU(N, \mathcal{D}(\theta^0))\theta_{ij} \geq \Omega_{ij} \quad \text{ for } t \in [t_0, T).$$

Then the standard Gronwall's lemma implies the desired estimate.
(ii) It follows from Lemma 3.2.1 that

$$\mathcal{D}(\theta(t)) \leq \mathcal{D}(\theta^0), \quad t \geq 0.$$

Let i, j be the indices such that

$$\theta_i(t_0) > \theta_j(t_0) \quad \text{ and } \quad \Omega_i > \Omega_j.$$

Then for such i and j, define a set $\mathcal{C}(i,j)$ and a number $T_{ij}^* \in [t_0, \infty]$:

$$\mathcal{C}(i,j) := \{T \in \mathbb{R}_+ \, : \, \theta_{ij}(t) > 0 \text{ and } \Omega_{ij} > 0, \, \forall\, t \in [t_0, T)\}, \quad T_{ij}^* := \sup \mathcal{C}(i,j).$$

Since $\theta_{ij}(t_0) > 0$ and $\Omega_{ij} > 0$, by the continuity of $\theta_{ij} = \theta_{ij}(t)$, there exists $T > t_0$ such that

$$\theta_{ij}(t) > 0, \quad t \in [t_0, T).$$

Of course, T may depend on $\theta_{ij}(t_0)$. Hence $T \in \mathcal{C}(i,j)$ which establishes that the set $\mathcal{C}(i,j)$ is not empty.

We now claim:

$$T_{ij}^* = \infty.$$

Suppose not, i.e., $T_{ij}^* < \infty$. Then we have

$$\lim_{t\to T_{ij}^*-} \theta_{ij}(t) = 0,$$

and on the time-interval $[t_0, T_{ij}^*)$, θ_{ij} is strictly positive, i.e.,

$$\theta_{ij}(t) > 0.$$

We now apply Lemma 3.3.1(i) to get

$$\theta_{ij}(t) \geq \theta_{ij}(t_0)e^{-KU(t-t_0)} + \frac{\Omega_{ij}}{KU}\big(1 - e^{-KU(t-t_0)}\big), \quad t \in [t_0, T_{ij}^*),$$

and let $t \to T_{ij}^*-$ to find

$$0 = \lim_{t \to T_{ij}^*-} \theta_{ij}(t) \geq \theta_{ij}(t_0)e^{-KU(T_{ij}^*-t_0)} + \frac{\Omega_{ij}}{KU}\Big(1 - e^{-KU(T_{ij}^*-t_0)}\Big) > 0,$$

which gives the contradiction. Hence $T_{ij}^* = \infty$ and we conclude that

$$\theta_{ij}(t) > 0, \qquad t \geq t_0.$$

7.2.2 The proof of Lemma 3.3.2

In this subsection, we provide the proof of Lemma 3.3.2.

(i) Suppose $\theta_{ij}^* < \theta_{ij}(t), \quad t \geq t_0$. Then since $\frac{\sin\theta_{ij}}{\theta_{ij}}$ is strictly decreasing function for $[0, \pi)$, we have

$$\frac{\sin\theta_{ij}(t)}{\theta_{ij}(t)} < \frac{\sin\theta_{ij}^*}{\theta_{ij}^*}, \quad t \geq t_0. \tag{7.2.3}$$

Then we use (7.2.3) and the equation (7.2.1) to find

$$\dot{\theta}_{ij} + KU\sin\theta_{ij} > \Omega_{ij} \quad \text{and} \quad KU\sin\theta_{ij}^* = \Omega_{ij}, \tag{7.2.4}$$

to derive the following Gronwall's inequality:

$$\dot{\theta}_{ij} + \frac{\Omega_{ij}}{\theta_{ij}^*}\theta_{ij} = \dot{\theta}_{ij} + KU\frac{\sin\theta_{ij}^*}{\theta_{ij}^*}\theta_{ij} > \dot{\theta}_{ij} + KU\sin\theta_{ij} > \Omega_{ij},$$

i.e.,

$$\dot{\theta}_{ij} + \frac{\Omega_{ij}}{\theta_{ij}^*}\theta_{ij} > \Omega_{ij}.$$

This yields

$$\theta_{ij}(t) > \theta_{ij}(t_0)e^{-\frac{\Omega_{ij}}{\theta^*_{ij}}(t-t_0)} + \theta^*_{ij}\Big(1 - e^{-\frac{\Omega_{ij}}{\theta^*_{ij}}(t-t_0)}\Big), \qquad t \geq t_0.$$

We now set

$$g(t) := \theta_{ij}(t_0)e^{-\frac{\Omega_{ij}}{\theta^*_{ij}}(t-t_0)} + \theta^*_{ij}\Big(1 - e^{-\frac{\Omega_{ij}}{\theta^*_{ij}}(t-t_0)}\Big).$$

Then it is easy to see that

$$\dot{g}(t) = \Omega_{ij}\Big(1 - \frac{\theta_{ij}(t_0)}{\theta^*_{ij}}\Big)e^{-\frac{\Omega_{ij}}{\theta^*_{ij}}(t-t_0)} < 0,$$

i.e, $g(t)$ is strictly decreasing, and $g(t)$ tends to θ^*_{ij} as $t \to \infty$.
Next we show that if there exists t_0 such that $\theta^*_{ij} < \theta_{ij}(t_0)$ then $\theta^*_{ij} < \theta_{ij}(t)$ for $t \geq t_0$. Since θ_{ij} is continuous function, there exists $T > 0$ such that

$$\theta^*_{ij} < \theta_{ij}(t), \quad \forall\, t \in [t_0, T).$$

We now define

$$\mathcal{S} := \{T > t_0 : \ \theta_{ij}(t) > \theta^*_{ij}, \quad \forall\, t \in [t_0, T)\}, \quad \text{and} \quad T^* := \sup \mathcal{S}.$$

Then clearly the set $\mathcal{S}$ is not empty. We next claim

$$T^* = \infty.$$

Suppose not, then we have

$$\theta^*_{ij} = \sin^{-1}\left(\frac{\Omega_{ij}}{KU}\right) = \lim_{t \to T^{*-}} \theta_{ij}(t) \geq \lim_{t \to T^{*-}} g(t) = g(T^*) > \lim_{t \to \infty} g(t) = \theta^*_{ij},$$

which gives the contradiction. Therefore, $T^* = \infty$ and we have

$$\theta_{ij}(t) > \theta^*_{ij}, \quad t \geq t_0.$$

(ii) We first show that A^+ and A^- have the same sign. By definition of $A^\pm$, we have

$$A^+A^- \;=\; \Big(\Omega_{ij}\tan\frac{\theta_{ij}}{2} - KU\Big)^2 - \lambda^2_{ij}$$

$$\begin{aligned}
&= \Omega_{ij}^2 \tan^2 \frac{\theta_{ij}}{2} - 2KU\Omega_{ij} \tan \frac{\theta_{ij}}{2} + \Omega_{ij}^2 \\
&= \Omega_{ij}^2 \sec^2 \frac{\theta_{ij}}{2} - 2KU\Omega_{ij} \tan \frac{\theta_{ij}}{2} \\
&= \Omega_{ij} \sec^2 \frac{\theta_{ij}}{2} \Big(\Omega_{ij} - KU \sin \theta_{ij} \Big) \\
&> 0,
\end{aligned}$$

where we used the fact that

$$\Omega_{ij} - KU \sin \theta_{ij} > \Omega_{ij} - KU \sin \theta_{ij}^* = 0.$$

We next to show that

$$\Omega_{ij} \tan \Big(\frac{\theta_{ij}}{2} \Big) - KU < 0,$$

which implies the negativity of A^-, so does A^+. By direct calculation, we have

$$\begin{aligned}
\Omega_{ij} \tan \frac{\theta_{ij}}{2} - KU &< \Omega_{ij} \tan \frac{\theta_{ij}^*}{2} - \frac{\Omega_{ij}}{\sin \theta_{ij}^*} \\
&= \Omega_{ij} \Big(\tan \frac{\theta_{ij}^*}{2} - \frac{1}{\sin \theta_{ij}^*} \Big) \\
&= \Omega_{ij} \Big(\frac{\sin(\frac{\theta_{ij}^*}{2})}{\cos(\frac{\theta_{ij}^*}{2})} - \frac{1}{2\cos(\frac{\theta_{ij}^*}{2}) \sin(\frac{\theta_{ij}^*}{2})} \Big) \\
&= \Omega_{ij} \Big(\frac{2\sin^2(\frac{\theta_{ij}^*}{2}) - 1}{\sin \theta_{ij}^*} \Big) \\
&= -\frac{\Omega_{ij}}{\tan \theta_{ij}^*} \\
&< 0,
\end{aligned}$$

where we used the elementary identity $1 - 2\sin^2(\frac{\theta_{ij}^*}{2}) = \cos \theta_{ij}^*$.

7.2.3 The proof of Proposition 3.3.1

In this subsection, we provide the proof of Proposition 3.3.1.

(i) We already showed that

$$KU\sin\theta_{ij} - \Omega_{ij} + \frac{d\theta_{ij}}{dt} > 0 \quad \text{in (7.2.4)}.$$

Since $\sin\theta_{ij} < \sin\theta^*_{ij} = \frac{\Omega_{ij}}{KU}$,

$$KU\sin\theta_{ij} - \Omega_{ij} < 0 \quad \text{and} \quad 1 + \frac{1}{KU\sin\theta_{ij} - \Omega_{ij}}\frac{d\theta_{ij}}{dt} < 0.$$

Since $KU > \Omega_{ij}$, by using the same method as in Lemma Appendix D.2, we can show that

$$\Big|\frac{\Omega_{ij}\tan(\frac{\theta_{ij}(t)}{2}) - KU + \lambda_{ij}}{\Omega_{ij}\tan(\frac{\theta_{ij}(t)}{2}) - KU - \lambda_{ij}}\Big| < e^{-C_1-\lambda_{ij}t}. \tag{7.2.5}$$

We use Lemma 3.3.2 (ii):

$$\Omega_{ij}\tan\Big(\frac{\theta_{ij}(t)}{2}\Big) - KU - \lambda_{ij} < \Omega_{ij}\tan\Big(\frac{\theta_{ij}(t)}{2}\Big) - KU + \lambda_{ij} < 0,$$

to find

$$C_1 < 0.$$

In (7.2.5), we have

$$-\Omega_{ij}\tan\Big(\frac{\theta_{ij}(t)}{2}\Big) + KU - \lambda_{ij} < e^{-C_1-\lambda_{ij}t}\Big[- \Omega_{ij}\tan\Big(\frac{\theta_{ij}(t)}{2}\Big) + KU + \lambda_{ij}\Big].$$

This again yields

$$\Omega_{ij}\tan\Big(\frac{\theta_{ij}(t)}{2}\Big) > \frac{KU - \lambda_{ij} - (KU+\lambda_{ij})e^{-C_1-\lambda_{ij}t}}{1 - e^{-C_1-\lambda_{ij}t}}, \quad t \geq t_*. \tag{7.2.6}$$

Here we used the fact that there exists t_* such that

$$e^{-C_1-\lambda_{ij}t} = e^{|C_1|-\lambda_{ij}t} < 1, \quad t \geq t_*.$$

(ii) We now take a limit $t \to \infty$ in (7.2.6) to find

$$\Omega_{ij}\tan\Big(\frac{1}{2}\lim_{t\to\infty}\theta_{ij}(t)\Big) \geq KU - \lambda_{ij}, \quad \text{i.e.,} \quad \lim_{t\to\infty}\theta_{ij}(t) \geq 2\tan^{-1}\Big(\frac{KU-\lambda_{ij}}{\Omega_{ij}}\Big). \tag{7.2.7}$$

On the other hand, if we follow exactly the same argument like in Lemma Appendix D.1, we have

$$2\tan^{-1}\left(\frac{KU-\lambda_{ij}}{\Omega_{ij}}\right) = \sin^{-1}\left(\frac{\Omega_{ij}}{KU}\right) = \theta^*_{ij}.$$

Hence we have

$$\lim_{t\to\infty}\theta_{ij}(t) \geq 2\tan^{-1}\left(\frac{KU-\lambda_{ij}}{\Omega_{ij}}\right) = \theta^*_{ij}. \tag{7.2.8}$$

On the other hand, it follows from the assumption that

$$\lim_{t\to\infty}\theta_{ij}(t) \leq \theta^*_{ij}. \tag{7.2.9}$$

Finally we combine (7.2.8) and (7.2.9) to conclude

$$\lim_{t\to\infty}\theta_{ij}(t) = \theta^*_{ij}.$$

7.3 Appendix C. Elementary estimates

In this section, we provide elementary estimates for the trigonometric functions and evaluation of certain integral involving with trigonometric for reader's convenience. These simple estimates are crucially used in the thesis.

Lemma 7.3.1. *The following estimate holds.*

$$\tan\left[\frac{1}{2}\sin^{-1}\left(\frac{\mathcal{D}(\Omega)}{K}\right)\right] = \frac{K-\lambda}{\mathcal{D}(\Omega)},$$

where λ is given by

$$\lambda := \sqrt{K^2 - (\mathcal{D}(\Omega))^2}. \tag{7.3.1}$$

Proof. For the simplicity of presentation, we set

$$\alpha := \sin^{-1}\left(\frac{\mathcal{D}(\Omega)}{K}\right), \quad \text{i.e.,} \quad \sin\alpha = \frac{\mathcal{D}(\Omega)}{K}, \quad \cos\alpha = \frac{\lambda}{K}.$$

It follows from (7.3.1) that

$$(\mathcal{D}(\Omega))^2 = (K-\lambda)(K+\lambda),$$

which is equivalent to

$$\frac{\mathcal{D}(\Omega)}{K+\lambda} = \frac{K-\lambda}{\mathcal{D}(\Omega)}. \tag{7.3.2}$$

On the other hand, by elementary properties of trigonometric functions and (7.3.2), we have

$$\tan\frac{\alpha}{2} = \frac{\sin\alpha}{2\cos\frac{\alpha}{2}} \cdot \frac{1}{\cos\frac{\alpha}{2}} = \frac{\sin\alpha}{\cos\alpha + 1} = \frac{\frac{\mathcal{D}(\Omega)}{K}}{1+\frac{\lambda}{K}} = \frac{\mathcal{D}(\Omega)}{K+\lambda} = \frac{K-\lambda}{\mathcal{D}(\Omega)},$$

where we used

$$\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}, \quad \cos\alpha + 1 = 2\cos^2\frac{\alpha}{2}.$$

$\square$

We now return to the differential equation:

$$\begin{aligned} &\frac{d\theta}{dt} = \Omega - K\sin\theta, \quad t > 0, \\ &\theta(0) = \theta^0. \end{aligned} \tag{7.3.3}$$

Since the equation (7.3.3) is separable, it can be written as an integral equation:

$$t = \int_{\theta^0}^{\theta(t)} \frac{d\theta}{\Omega - K\sin\theta}. \tag{7.3.4}$$

We next evaluate the following integral:

$$I(K,\Omega) := \int_{\theta^0}^{\theta(t)} \frac{d\theta}{\Omega - K\sin\theta}. \tag{7.3.5}$$

By direct calculation, we have

$$\begin{aligned} I(K,\Omega) &= \int_{\theta^0}^{\theta(t)} \frac{d\theta}{\Omega - 2K\sin\frac{\theta}{2}\cos\frac{\theta}{2}} \\ &= \int_{\theta^0}^{\theta(t)} \frac{\Omega\sec^2\frac{\theta}{2}d\theta}{\Omega^2\sec^2\frac{\theta}{2} - 2K\Omega\tan\frac{\theta}{2}} \\ &= \int_{\theta^0}^{\theta(t)} \frac{\Omega\sec^2\frac{\theta}{2}d\theta}{\Omega^2(1+\tan^2\frac{\theta}{2}) - 2K\Omega\tan\frac{\theta}{2}} \\ &= \int_{\theta^0}^{\theta(t)} \frac{\Omega\sec^2\frac{\theta}{2}d\theta}{(\Omega\tan\frac{\theta}{2} - K)^2 + \Omega^2 - K^2}. \end{aligned} \tag{7.3.6}$$

In the following lemma, we explicitly proceed the evaluation of the integral $I(K,\Omega)$ and find the explicit solution to the equation (7.3.3).

Lemma 7.3.2. *The following estimates hold.*
Case 1 ($K > |\Omega|$):

$$t\sqrt{K^2-\Omega^2} = \log\Big|\frac{\Omega\tan\frac{\theta(t)}{2} - K - \sqrt{K^2-\Omega^2}}{\Omega\tan\frac{\theta(t)}{2} - K + \sqrt{K^2-\Omega^2}}\Big| - \log\Big|\frac{\Omega\tan\frac{\theta^0}{2} - K - \sqrt{K^2-\Omega^2}}{\Omega\tan\frac{\theta^0}{2} - K + \sqrt{K^2-\Omega^2}}\Big|.$$

Case 2 ($K = |\Omega|$):

$$t = \frac{2}{\Omega\tan\frac{\theta^0}{2} - K} - \frac{2}{\Omega\tan\frac{\theta(t)}{2} - K}.$$

Case 3 ($K < |\Omega|$)*:*

$$\tan\frac{\theta(t)}{2} = \frac{1}{R^\infty}\Big\{\sqrt{(R^\infty)^2-1}\tan\Big[\frac{Kt}{2}\sqrt{(R^\infty)^2-1}+\tan^{-1}\Big(\frac{R^\infty\tan\frac{\theta^0}{2}-1}{\sqrt{R^\infty-1}}\Big)\Big]+1\Big\}.$$

Proof. Since the evaluation of the integral will depend on the sign of $\Omega^2 - K^2$, we will need to consider the following three cases:

$$K > |\Omega|, \quad K = |\Omega| \quad \text{and} \quad K < |\Omega|.$$

Case 1 ($K > |\Omega|$): In (7.3.6), we have

$$\begin{aligned}
I(K,\Omega) &= \int_{\theta^0}^{\theta(t)} \frac{\Omega\sec^2\frac{\theta}{2}}{(\Omega\tan\frac{\theta}{2}-K)^2-(K^2-\Omega^2)}d\theta \\
&= 2\int_{\theta^{0*}}^{\theta^*(t)} \frac{1}{\alpha^2-(\sqrt{K^2-\Omega^2})^2}d\alpha \\
&= 2\int_{\theta^{0*}}^{\theta^*(t)} \frac{1}{(\alpha+\sqrt{K^2-\Omega^2})(\alpha-\sqrt{K^2-\Omega^2})}d\alpha \\
&= \frac{1}{\sqrt{K^2-\Omega^2}}\int_{\theta^{0*}}^{\theta^*(t)}\Big(\frac{1}{\alpha-\sqrt{K^2-\Omega^2}}-\frac{1}{\alpha+\sqrt{K^2-\Omega^2}}\Big)d\alpha \\
&= \frac{1}{\sqrt{K^2-\Omega^2}}\Big(\log\Big|\frac{\Omega\tan\frac{\theta(t)}{2}-K-\sqrt{K^2-\Omega^2}}{\Omega\tan\frac{\theta(t)}{2}-K+\sqrt{K^2-\Omega^2}}\Big| \\
&- \log\Big|\frac{\Omega\tan\frac{\theta^0}{2}-K-\sqrt{K^2-\Omega^2}}{\Omega\tan\frac{\theta^0}{2}-K+\sqrt{K^2-\Omega^2}}\Big|\Big),
\end{aligned}$$

where θ^{0*} and $\theta^*(t)$ are given by the following relations:

$$\theta^{0*} := \Omega\tan\frac{\theta^0}{2}-K, \qquad \theta^*(t) := \Omega\tan\frac{\theta(t)}{2}-K.$$

Hence in (7.3.4), we have

$$t\sqrt{K^2-\Omega^2} = \log\Big|\frac{\Omega\tan\frac{\theta(t)}{2}-K-\sqrt{K^2-\Omega^2}}{\Omega\tan\frac{\theta(t)}{2}-K+\sqrt{K^2-\Omega^2}}\Big|-\log\Big|\frac{\Omega\tan\frac{\theta^0}{2}-K-\sqrt{K^2-\Omega^2}}{\Omega\tan\frac{\theta^0}{2}-K+\sqrt{K^2-\Omega^2}}\Big|.$$

Case 2 ($K = |\Omega|$): In (7.3.6), we have

$$I(K,\Omega) = \int_{\theta^0}^{\theta(t)} \frac{\Omega\sec^2\frac{\theta}{2}}{(\Omega\tan\frac{\theta}{2}-K)^2}d\theta$$

$$\begin{aligned} &= 2\int_{\theta^{0*}}^{\theta^*(t)} \frac{1}{\alpha^2} d\alpha \\ &= -2\Big(\frac{1}{\Omega\tan\frac{\theta(t)}{2} - K} - \frac{1}{\Omega\tan\frac{\theta^0}{2} - K}\Big), \end{aligned}$$

where θ^{0*} and $\theta^*(t)$ are given by the following relations:

$$\theta^{0*} := \Omega\tan\frac{\theta^0}{2} - K, \qquad \theta^*(t) := \Omega\tan\frac{\theta(t)}{2} - K.$$

Hence we have

$$t = \frac{2}{\Omega\tan\frac{\theta^0}{2} - K} - \frac{2}{\Omega\tan\frac{\theta(t)}{2} - K}.$$

Case 3 ($K < |\Omega|$): In this case, since $\Omega^2 - K^2 > 0$, it follows from (7.3.6) that

$$\begin{aligned} I(K,\Omega) &= \frac{1}{\Omega^2 - K^2}\int_{\theta^0}^{\theta(t)} \frac{\Omega\sec^2\frac{\theta}{2}}{\Big(\frac{\Omega\tan\frac{\theta}{2} - K}{\sqrt{\Omega^2-K^2}}\Big)^2 + 1} d\theta \\ &= \frac{2}{\sqrt{\Omega^2 - K^2}}\int_{\theta^{0*}}^{\theta^*(t)} \frac{d\alpha}{\alpha^2+1}, \end{aligned}$$

where θ^{0*} and θ^* are given by the following formula:

$$\theta^{0*} = \frac{\Omega\tan\frac{\theta^0}{2} - K}{\sqrt{\Omega^2 - K^2}}, \quad \theta^*(t) = \frac{\Omega\tan\frac{\theta(t)}{2} - K}{\sqrt{\Omega^2 - K^2}}.$$

We continue the evaluation of the integral.

$$\begin{aligned} I(K,\Omega) &= \frac{2}{\sqrt{\Omega^2 - K^2}}\Big[\tan^{-1}\theta^*(t) - \tan^{-1}\theta^{0*}\Big] \\ &= \frac{2}{\sqrt{\Omega^2 - K^2}}\Big[\tan^{-1}\Big(\frac{\Omega\tan\frac{\theta(t)}{2} - K}{\sqrt{\Omega^2 - K^2}}\Big) - \tan^{-1}\Big(\frac{\Omega\tan\frac{\theta^0}{2} - K}{\sqrt{\Omega^2 - K^2}}\Big)\Big] \\ &= \frac{2}{K\sqrt{(R^\infty)^2 - 1}}\Big[\tan^{-1}\Big(\frac{R^\infty\tan\frac{\theta(t)}{2} - 1}{\sqrt{(R^\infty)^2 - 1}}\Big) - \tan^{-1}\Big(\frac{R^\infty\tan\frac{\theta^0}{2} - 1}{\sqrt{(R^\infty)^2 - 1}}\Big)\Big], \end{aligned}$$

where $R^\infty = \frac{\Omega}{K}$. Hence it follows from (7.3.4) to get

$$t = \frac{2}{K\sqrt{(R^\infty)^2 - 1}}\Big[\tan^{-1}\Big(\frac{R^\infty\tan\frac{\theta(t)}{2} - 1}{\sqrt{(R^\infty)^2 - 1}}\Big) - \tan^{-1}\Big(\frac{R^\infty\tan\frac{\theta^0}{2} - 1}{\sqrt{(R^\infty)^2 - 1}}\Big)\Big].$$

This can be rewritten as

$$\frac{Kt}{2}\sqrt{(R^{\infty})^2-1}+\tan^{-1}\left(\frac{R^{\infty}\tan\frac{\theta^0}{2}-1}{\sqrt{(R^{\infty})^2-1}}\right)=\tan^{-1}\left(\frac{R^{\infty}\tan\frac{\theta(t)}{2}-1}{\sqrt{(R^{\infty})^2-1}}\right).$$

We rearrange the above relation again to find

$$\tan\frac{\theta(t)}{2}=\frac{1}{R^{\infty}}\left\{\sqrt{(R^{\infty})^2-1}\tan\left[\frac{Kt}{2}\sqrt{(R^{\infty})^2-1}+\tan^{-1}\left(\frac{R^{\infty}\tan\frac{\theta^0}{2}-1}{\sqrt{(R^{\infty})^2-1}}\right)\right]+1\right\}.$$

$\square$

Bibliography

[1] Acebron, J. A., Bonilla, L. L., Vicente, C.J.P. Pérez, Ritort, F., Spigler, R. :*The Kuramoto model: A simple paradigm for synchronization phenomena.* Rev. Mod. Phys. **77** 137-185 (2005).

[2] Acebrón, J. A., Bonilla, L. L., Spigler, R.: *Synchronization in populations of globally coupled oscillators with inertial effect.* Phys. Rev. E **62**, 3437 (2000).

[3] Artstein, Z., Kevrekidis, I. G., Slemrod, M. and Titi, E. S.: *Slow observables of singularly perturbed differential equations.* Nonlinearity **20**, 2463-2481 (2007).

[4] Aoki, I.: *A simulation study on the schooling mechanism in fish.* Bulletin of the Japan Society of Scientific Fisheries. **48**, 1081-1088 (1982).

[5] Artstein, Z.: *On singularly perturbed ordinary differential equations with measure-valued limits.* Math. Bohem. **127**, 139-152 (2002).

Prog. Theor. Phys. 921-941 (2004).

[6] Ashwin, P., Swift, J. W.: *The dynamics of n weakly coupled identical oscillators.* J. Nonlin. Sci. **2** 69-108(1992).

[7] Artstein, Z. and Vigodner, A.: *Singularly perturbed ordinary differential equations with dynamic limits.* Proc. Roy. Soc. Edinburgh Sect. A **126**, 541-569 (1996).

[8] Ball, J. M.: *A version of the fundamental theorem for Young measures.* Partial Differential Equations and Continuum Models of Phase

Transitions. Lecture notes in Physics **344**, 207-215 (1989) edited by M. Rascle et al. Berlin Springer.

[9] Billingsley, P.: *Convergence of probability measures.* New York Wiley 1968.

[10] Carrillo, J. A., Fornasier, M., Rosado, J. and Toscani, G.: *Asymptotic flocking dynamics for the kinetic Cucker-Smale model.* SIAM J. Math. Anal. **42**, 218-236 (2010).

[11] Chuang, Y.-L., D'Orsogna, M. R., Marthaler, D., Bertozzi, A. L., Chayes, L.: *State transitions and the continuum limit for a 2D interacting self-propelled particle system.* Physica D, **232**, 33-47 (2007).

[12] Carrillo, J. A., D'Orsogna, M. R. D. and Panferov, V.: *Double milling in self-propelled swarms from kinetic theory.* Kinetic and Related Models **2**, 363-378 (2009).

[13] Choi, Y.-P., Ha, S.-Y., Jung, S.E. and Kim, Y.: *Asymptotic formation and orbital stability of phase-locked states for the Kuramoto model.* Physica D **241**, 735-754 (2012).

[14] Choi, Y.-P., Ha, S.-Y., Yun, S.-B. : *Complete synchronization of Kuramoto oscillators with finite inertia* Physica D **240** 32-44 (2011).

[15] Coddington, E.A. and Levinson, N,: *Theory of Ordinary Differential Equations.* McGraw-Hill book Company, Inc. New York Toronto London (1955).

[16] Chiba, H., Nishikawa, I.: *Center manifold reduction for a large population of globally coupled phase oscillators.* Submitted (2011).

[17] Chiba, H., Pazo, D.: *Stability of an* $[N/2]$*-dimensional invariant torus in the Kuramoto model at small coupling.* Physica D **238** 1068-1081 (2009).

[18] Chopra, N. and Spong,M.W. : *On exponential synchronization of Kuramoto oscillators.* IEEE Trans. Automatic Control **54**, 353-357(2009).

[19] Cucker, F. and Smale, S.: *On the mathematics of emergence*. Japan. J. Math. **2**, 197-227 (2007).

[20] Cucker, F. and Smale, S.: *Emergent behavior in flocks*. IEEE Trans. Automat. Control **52**, 852-862 (2007).

[21] De Smet, F., Aeyels, D. : *Partial frequency in finite Kuramoto-Sakaguchi model.* Physica D 81-89 (2007).

[22] Dorfler, F., Bullo, F.: *On the critical coupling for Kuramoto oscillators.* To appear in SIAM. J. Appl. Dynamical.

[23] Daniels, B. C., Dissanayake, S. T. and Trees, B. R.: *Synchronization of coupled rotators: Josephson junction ladders and the locally coupled Kuramoto model.* Phys. Rev. E. **67**, 026216 (2003).

[24] Degond, P. and Motsch, S.: *Macroscopic limit of self-driven particles with orientation interaction.* C.R. Math. Acad. Sci. Paris **345**, 555-560 (2007).

[25] Degond, P. and Motsch, S.: *Large-scale dynamics of the persistent Turing Walker model of fish behavior.* J. Stat. Phys. **131**, 989-1022 (2008).

[26] Degond, P. and Motsch, S.: *Continuum limit of self-driven particles with orientation interaction.* Math. Mod. Meth. Appl. Sci. **18**, 1193-1215 (2008).

[27] D'Orsogna, M. R., Chuang, Y. L., Bertozzi, A. L. and Chayes, L.: *Self-propelled particles with soft-core interactions: patterns, stability and collapse.* Phys. Rev. Lett. **96**,104302 (2006).

[28] Dunkel, J., Ebeling, W. and Erdmann, U.: *Theormodynamics and transport in an active Morse ring chain.* Eur. Phys. J. B **24**, 511-524 (2001).

[29] Ermentrout, G.B.: *Synchronization in a pool of mutually coupled oscillators with random frequencies.* J. Math. Biol. **22** 1-9 (1985).

[30] Ha, S.-Y., Ha, T. and Kim, J.-H.: *Asymptotic flocking dynamics for the Cucker-Smale mdoel with the Rayleigh friction.* J. Physics A: Math. Theor. **43**, 315201 (2010).

[31] Ha, S.-Y., Ha, T., Kim, J.-H. :*On the complete synchronization of the Kuramoto phase model* Physica D **239** 1692-1700 (2010).

[32] Ha, S.-Y., Jung, S. and Slemrod, M.: *Fast-slow dynamics of planar particle models for flocking and swarming.* J. Differential Equations **252**, 2563-2579 (2012).

[33] Ha, S.-Y., Lattanzio, C., Rubino, B. and Slemrod, M.: *Flocking and synchronization of particle models.* Quarterly of Applied Mathematics, **69**, 91-103 (2011).

[34] Ha, S.-Y. and Liu, J.-G.: *A simple proof of Cucker-Smale flocking dynamics and mean field limit*. Commun. Math. Sci. **7**, 297-325 (2009).

[35] Ha, S.-Y. and Slemrod, M.: *A fast-slow dynamical systems theory for the Kuramoto type phase model*, To appear in J. Differential Equations.

[36] Ha, S.-Y. and Slemrod, M.: *Flocking dynamics of singularly perturbed oscillator chain and the Cucker-Smale system.* J. Dyn. Diff. Equat. **22**, 325-330 (2010).

[37] Ha, S.-Y. and Tadmor, E.: *From particle to kinetic and hydrodynamic description of flocking*. Kinetic and Related Models. **1**, 415-435 (2008).

[38] Hale, J. K.: *Ordinary differential equations.* Krieger (1980).

[39] Hale, J. K. :*Ordinary differential equations.* Dover Publications, 1991.

[40] Hong, H., Choi, M. Y., Yi, J. and Soh, K.-S.: *Inertial effects on periodic synchronization in a system of coupled oscillators.* Phys. Rev. E. **59**, 353-363 (1999).

[41] Ha, S.-Y., Ha, T. and Kim, J.-H.: *Asymptotic flocking dynamics for the Cucker-Smale mdoel with the Rayleigh friction.* J. Physics A: Math. Theor. **43**, 315201 (2010).

[42] Hong, H., Jeon, G. S. and Choi, M. Y.: *Spontaneous phase oscillation induced by inertia and time delay.* Phys. Rev. E. **65**, 026208 (2002).

[43] Ha, S.-Y., Jung, S. and Slemrod, M.: *Fast-slow dynamics of planar particle models for flocking and swarming.* J. Differential Equations **252**, 2563-2579 (2012).

[44] Ha, S.-Y., Slemrod M.: *A fast-slow dynamical systems theory for the Kuramoto phase model.* J. Differential Equations. **251**, 1-25 (2011).

[45] Kuramoto, Y. and Nishikawa, I.: *Onset of collective rhythms in large populations of coupled oscillators* in Cooperative dynamics in complex physical systems. Edited by H. Takayama. 300-306 (1988).

[46] Kuramoto, Y. : *Chemical Oscillations, waves and turbulence.* Springer-Verlag Berlin (1984).

[47] Kuramoto, Y. : *International symposium on mathematical problems in mathematical physics.* Lecture notes in theoretical physics. **30** 420 (1975).

[48] Levi, M., Hoppensteadt, F. C. and Miranker, W. L.: *Dynamics of the Josephson junction.* Quarterly of Appl. Math. **36**, 167-198 (1978/79).

[49] Li, Y.-X., Lukeman, R. and Edelstein-Keshet, L.: *Minimal mechanisms for school formation in self-propelled particles.* Physica D **237**, 699-720 (2008).

[50] Lukeman, R., Li, Y.-X. and Edelstein-Keshet, L.: *Inferring individual rules from collective behavior.* PNAS **107**, 12576-12580 (2010).

[51] Lukeman, R., Li, Y.-X. and Edelstein-Keshet, L.: *A conceptual model for milling formations in biological aggregates.* Bull. Math. Biol. **71**, 352-382 (2008).

[52] Maistrenko, Y.L, Popovych, O. V. , Burylko, O., Tass, P. A.: *Mechanism of Desynchronization in the Finite-Dimensional Kuramoto Model* Phys. Review Lett. **93** 084102 (2004).

[53] Maistrenko, Y.L., Popovych, O.V., Tass, P.A. :*Desynchronization and Chaos in the Kuramoto Model.* Lecture Notes in Physics, **671** 285-306 (2005).

[54] Nemytskii, V. V. and Stepanov, V. V.: *Qualitative theory of differential equations.* Princeton University Press, 1960.

[55] O'Malley, R. E.: *Singular perturbation methods for ordinary differential equations.* Applied Mathematical Sciences, **89**, Springer-Verlag, New York 1991.

[56] Park, K., Choi, M. Y.: *Synchronization in networks of superconducting wires.* Phys. Rev. B **56**, 387-394 (1997).

[57] Sanders, J. A., Verhulst, F. and Murdock J.: *Averaging methods in nonlinear dynamical systems.* Springer-Verlag, New York. 1985.

[58] Schweitzer, F., Ebeling, W. and Tilch, B.: *Statistical mechanics of canonical-dissipative systems and applications to swarm dynamics.* Phys. Review E. **64**, 021110 (2001).

[59] Slemrod, M.: *Averaging of fast-slow systems* in Lecture Notes in Computational Science and Engineering, Vol. **75**, ed. Alexander N. Gorban and Dirk Roose, 2011.

[60] Sumpter, D.J.T., Krause, J. James, R., Couzin, I.D. and Ward, A.J.W.: *Consensus decision-making by fish.* Current Biology, **105**, 6948-6953 (2008).

[61] Stoker, J. J.: *Nonlinear vibrations in mechanical and electrical systems.* Interscience, N. Y. 1950.

[62] Tartar, L.: *Compensated compactness and applications to partial differential equations.* In Research Notes in Mathematics **39**, Nonlinear

analysis and mechanics: Hariot-Watt Symposium, Vol. **4**, ed. R.J. Knops, Pittman Press, 136-211 (1975).

[63] Tikhonov, A. N.: *Systems of differential equations containing a small parameter multiplying the derivative.* Math. Sb. N.S. **27**, 147-156 (1950).

[64] Topaz, C. M. and Bertozzi, A. L.: *Swarming patterns in a two-dimensional kinematic model for biological groups.* SIAM J. Appl. Math.**65**, 152-174 (2004).

[65] Tanaka, H. A., Lichtenberg, A. J. and Oishi, S.: *First order phase transitions resulting from finite inertia in coupled oscillators systems.* Phys. Rev. Lett. **78**, 2104 (1997).

[66] Tanaka, H. A., Lichtenberg, A. J. and Oishi, S.: *Self-synchronization of coupled oscillators with hysteretic responses.* Physica D **100**, 279-300 (1997).

[67] Valadier, M.: *A course on Young meaures.* Rend. Istit. Mat. Univ. Trieste **26** (Suppl.), 349-394 (1994).

[68] Vicsek, T., Czirók, Ben-Jacob, E., Cohen, I. and Schochet, O.: *Novel type of phase transition in a system of self-driven particles.* Phys. Rev. Lett. **75**, 1226-1229 (1995).

[69] Wiesenfeld, K. , Colet, R. and Strogatz, S. H.: *Synchronization transitions in a disordered Josephson series arrays.* Phys. Rev. Lett. **76**, 404-407 (1996).

[70] Wiesenfeld, K. , Colet, R. and Strogatz, S. H.: *Frequency locking in Josephson arrays: connection with the Kuramoto model.* Phys. Rev. E **57**, 1563-1569 (1988).

[71] Ward, A.J.W., Herbert-Read, J., Sumpter, D.J.T. and Krause, J.: *Fast and accurate decisions through collective vigilance in fish shoals.* PNAS (2011).

[72] Winfree, A.T. *The Geometry of biological time.* Springer New York (1980).

[73] Watanabe, S., Strogatz, S. H.: *Constants of motion for superconducting Josephson arrays.* Physica D **74** 197-253(1994).

[74] Wiesenfeld, K., Swift, J.W.: *Averaged equations for Josephson junction series arrays.*Phys. Review E **51** 1020-1025(1995).

국문초록

이 논문에서는, 우리는 플로킹과 동기화 모델들에 대해 연구한다. 특히, 쿠라모토 모델, 관성이 있는 쿠라모토 모델, 쿠커-스매일 모델과 레일레이 마찰이 있는 뉴턴 타입 모델을 다룬다. 우리는 특이 극한에서의 역학계의 질적인 기술을 유도하기 위하여 알스테인- 케브리키디스-슬렘로드- 티티의 특이 섭동을 위한 통일된 접근범을 이용한다. 제 2장에서는 알스테인-케브리키디스- 슬렘로드-티티의 특이 섭동 이론과 평면에서의 포앙카레-벤딕슨 이론을 제시한다. 제 3장에서는 비동등 쿠라모토 진동자의 앙상블로부터 유도된 위상 동기 상태의 점근적 형성을 논의한다. 위상 동기 상태의 형성 과정에서, 우리는 진동자 사이의 진동 수와 횡단 위상 차의 하한-상한을 추정한다. 제 4장에서, 우리는 관성이 있는 쿠라모토 타입 모델을 위한 패스트-슬로우 역학 시스템을 제시한다. 우리의 새로운 형태에서, 순서 매개변수는 특이 섭동의 알스테인-케브리키디스-슬렘로드-티티의 이론의 체계에서 직각의 하측치로서의 역할을 한다. 제 5장에서 우리는 평면에서의 입자모델(쿠커-스매일 모델과 레일레이 마찰이 있는 뉴턴 타입 모델)의 패스트- 슬로우 역학계를 논의한다. 우리의 분석은 의사소통 하중의 최소한의 가정을 사용한다. 제 6장에서는 플로킹 모델과 관련된 수학적 문제를 간단히 제시하고 이 논문을 요약한다.

주요어휘:

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